MATH 592 1/24/2024

**Theorem**: $X = U \cup V$, $U, V \subseteq X$ open. Then there is a pushout of groupoids

$$\begin{tikzcd} \Pi(U \cap V) \arrow[r] \arrow[d] & \Pi(U) \arrow[d] \\ \Pi(V) \arrow[r] & \Pi(X) \end{tikzcd}.$$

**Proof**: Prove universality. Groupoid $\Gamma$

$$\begin{tikzcd} \Pi(U \cap V) \arrow[r] \arrow[d] & \Pi(U) \arrow[d] \arrow[ddr, bend left, "f_1"] & \\ \Pi(V) \arrow[r] \arrow[drr, bend right, "f_2"'] & \Pi(X) \arrow[dr, dashed] & \\ & & \Gamma \end{tikzcd}$$

$\omega : [0,1] \to X$

$\exists N \quad \omega\left[\frac{k}{N}, \frac{k+1}{N}\right] \subseteq U$ or $V$

set $\varepsilon_k = 0$    set $\varepsilon_k = 1$

$\omega = \omega_0 * \omega_1 * \cdots * \omega_{N-1}$

$\pi(X)$

$\ell_k : [\frac{k}{N}, \frac{k+1}{N}] \longrightarrow [0,1]$

linear increasing bijection

$\omega_k(t) = \omega(\ell_k^{-1}(t))$

We must send $\omega \longmapsto f_{\varepsilon_0}(\omega_0) \cdot f_{\varepsilon_1}(\omega_1) \cdot \cdots \cdot f_{\varepsilon_{N-1}}(\omega_{N-1})$

This proves uniqueness.

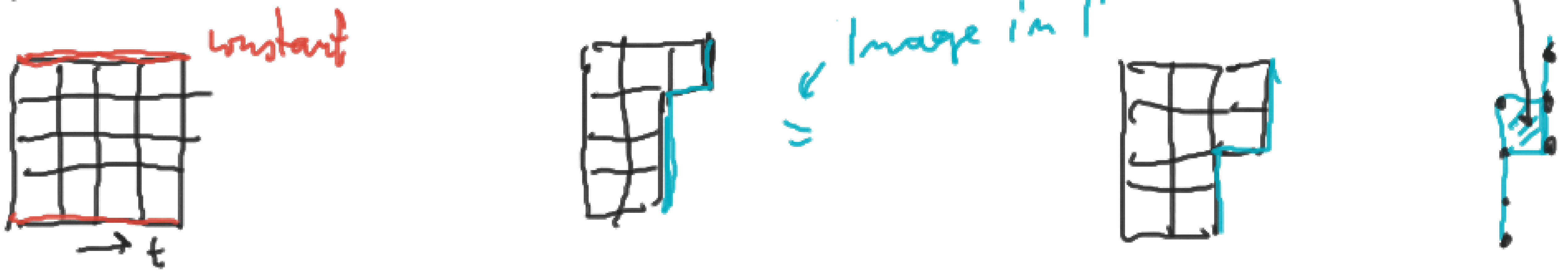
To prove existence, consider a homotopy $h : [0,1] \times [0,1] \longrightarrow X$

t, the homotopy coordinate

$h_t(0)$ constant in $t$, as is $h_t(1)$.

in $U$ or in $V$

$\exists N \quad h\Big|_{[\frac{k}{N}, \frac{k+1}{N}] \times [\frac{l}{N}, \frac{l+1}{N}]} \subseteq U$ or $V$




$\square$

What does the pushout of groupoids look like?

We considered $\pi(X)$ up to equivalence of categories.

Key question: does the pushout of groupoids preserve equivalence of categories?

Answer is: <u>yes</u> for a pushout

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{g_1} & \Gamma_2 \\ {\scriptstyle g_2}\downarrow & & \\ \Gamma_3 & & \end{array}$$

if $g_1, g_2$ are injective on objects.

This statement can be used to calculate $\pi_1(S^1)$

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We concentrate on the case of path-connected spaces.

$\Gamma_i$ is connected

**Lemma:** Let

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{g_1} & \Gamma_2 \\ {\scriptstyle g_2}\downarrow & & \downarrow \\ \Gamma_3 & \longrightarrow & \Phi \end{array}$$

be a pushout of groupoids where all objects of $\Gamma_i$ are isomorphic for each $i = 1,2,3$

Choose $x \in \mathrm{Obj}\,\Gamma_1$. Assume $g_1, g_2$ are injective on objects. Then

$$(\dagger)\qquad \begin{tikzcd}
\mathrm{Aut}_{\Gamma_1}(x) \arrow[r, "g_1"] \arrow[d, "g_2"'] & \mathrm{Aut}_{\Gamma_2}(x) \arrow[d] \\
\mathrm{Aut}_{\Gamma_3}(x) \arrow[r] & \mathrm{Aut}_{\Phi}(x)
\end{tikzcd}$$

is a pushout of groups.

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Proof: Prove that $(\dagger)$ is a pushout of groups. Existence:

$$(*)\qquad \begin{tikzcd}
\Gamma_1 \arrow[rrr] \arrow[dd] & & & \Gamma_2 \arrow[ddd, bend left] \\
\mathrm{Aut}_{\Gamma_1}(x) \arrow[u, phantom, sloped, "\subset"] \arrow[rr] \arrow[d] & & \mathrm{Aut}_{\Gamma_2}(x) \arrow[ur, phantom, sloped, "\subset"] \arrow[dd] & \\
\Gamma_3 \arrow[drrr] & & & \\
\mathrm{Aut}_{\Gamma_3}(x) \arrow[u, phantom, sloped, "\subset"] \arrow[rrr] & & & G
\end{tikzcd}$$

group

Strategy: define this defining diagram.

$$\begin{tikzcd} \text{Object } x \quad \Gamma_1 \arrow[r, "g_1"] \arrow[d, "g_2"'] & \Gamma_2 \\ \Gamma_3 & \end{tikzcd}$$

Complete a diagram

For every $y \in \mathrm{Obj}\,\Gamma_1$ select an $\cong$ $y \xrightarrow[\gamma_y]{\cong} x$ $\in \Gamma_1$ (select $\gamma_x = \mathrm{Id}$).

Injectivity on objects of $g_1, g_2$: these isomorphisms are also valid in $\Gamma_2, \Gamma_3$.

Of course, there are additional objects $y \in \Gamma_i$, $i = 2,3$.

Again, select $\gamma_y : y \xrightarrow{\cong} x$ in $\Gamma_2, \Gamma_3$ (whichever applies).

$$\begin{tikzcd} \Gamma_1 \arrow[r, "g_1"] \arrow[d, "g_2"'] & \Gamma_2 \arrow[d] \arrow[dd, bend left] \\ \Gamma_3 \arrow[r] \arrow[dr, bend right] & \Phi \arrow[d, dashed] \\ & G \end{tikzcd}$$

by sending all $\gamma_y \mapsto \mathrm{Id}$

Existence follows from restricting to $\mathrm{Aut}_{\Phi}(x)$.

## Uniqueness: Similar

$$\begin{tikzcd}
\mathrm{Aut}_{\Gamma_1}(x) \arrow[r, "g_1"] \arrow[d, "g_2"'] & \mathrm{Aut}_{\Gamma_2}(x) \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{Aut}_{\Gamma_3}(x) \arrow[r] \arrow[drr, bend right] & \mathrm{Aut}_{\Phi}(x) \arrow[dr, shift left, "f_1"] \arrow[dr, shift right, "f_2"'] & \\
 & & G
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma_1 \arrow[r] \arrow[d] & \Gamma_2 \arrow[d] \arrow[ddr, bend left] & \\
\Gamma_3 \arrow[r] \arrow[drr, bend right] & \Phi \arrow[dr, "f_1"] & \\
 & & G
\end{tikzcd}$$

$f_1 \neq f_2$ which make this commute.

Using the above setup $\Gamma_4$, you can construct two different homomorphisms of groupoids, violating the uniqueness for $\Phi$

<u>Small observation:</u>

$$\begin{tikzcd}
\mathrm{Obj}\,\Gamma_1 \arrow[r] \arrow[d] & \mathrm{Obj}\,\Gamma_2 \arrow[d] \\
\mathrm{Obj}\,\Gamma_3 \arrow[r] & \mathrm{Obj}\,\Phi
\end{tikzcd}$$

is a pushout of sets.

In particular, $\Phi$ is connected. $\square$

## What does pushout of groups look like?

**Does it even exist?** (YES)

Start with a pushout of groups of the form

$$\begin{tikzcd} \{e\} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\cap" rotate=90] & G \arrow[d, "i"] \\ H \arrow[r, "j"'] & G * H \end{tikzcd}$$

the free product

In other words, $g : G \to \Gamma$

$h : H \to \Gamma$

homomorphism of groups

$\exists!\, f : G * H \to \Gamma$ $\quad f \circ i = g$

$f \circ j = h.$

This is called a (categorical) coproduct. Categorically denoted by $G \amalg H$. Similarly: $\coprod_{s \in S} G_s$

$*_s G_s$

$$\underbrace{*_S \mathbb{Z}}_{\text{coproduct over a set } S \text{ of copies of } \mathbb{Z}} = FS \longleftarrow \text{The free group on a set } S$$

We have a map of sets $S \xrightarrow{i} FS$

<u>Universal property:</u> For every group $G$ and every map of sets $f: S \to G$ there exists a unique homomorphism of groups $F: FS \to G$ such that $F\,i(s) = f(s)$.

$U, V \subseteq X$ open
$U, V, U \cap V$ path-connected, $x \in U \cap V$

$$\begin{tikzcd}
\pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(U, x) \arrow[d] \\
\pi_1(V, x) \arrow[r] & \pi_1(X, x)
\end{tikzcd}$$

is a pushout

The group $FS$ can be constructed as

$$\{\underbrace{x_1 \dots x_n}_{\text{word}} \mid x_i \text{ is an element } s \in S \text{ or } s^{-1}, s \in S,\ n \in \mathbb{N}_0\} / \sim$$

$\sim$: smallest equivalence relation containing

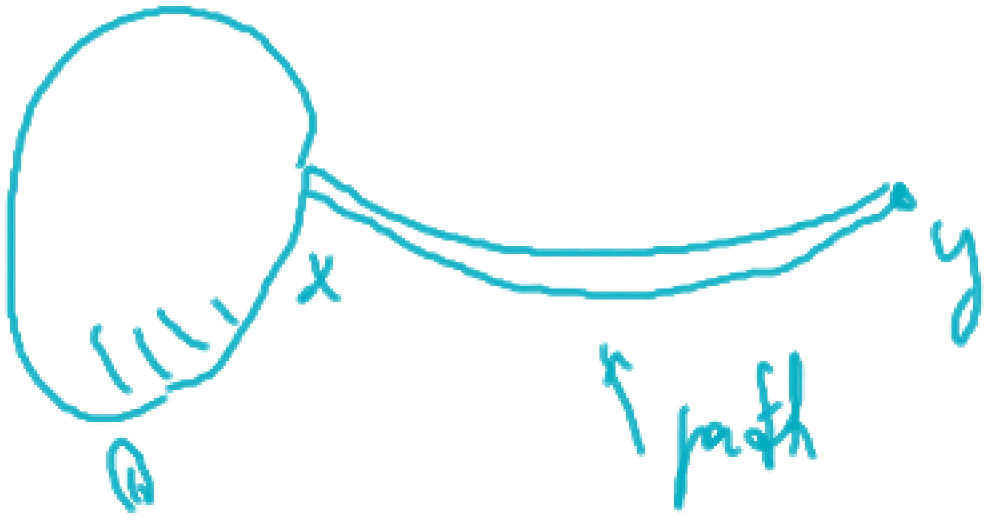
$w s s^{-1} w' \sim w w'$

$w s^{-1} s w' \sim w w'$

$w, w'$ words.

HW ④ Construct a functor $\Pi_n : \Pi(X) \to Ab$, $n \geq 2$ ← abelian groups, homo.

which on objects $x \longmapsto \pi_n(X, x)$.




$n = 2$ $\pi_2(X, x)$

$\omega : [0,1] \to X$

$\omega(0) = x \quad \omega(1) = y$

$\pi_n(X, x) \longrightarrow \pi_n(X, y)$

dependency on $[\omega] \in \Pi(X)$