MATH 592 1/26/2024

Let S be a set. The free group on S:

$$FS = \{\underbrace{x_1 \dots x_n}_{\text{words}} \mid x_i = s \text{ or } x_i = s^{-1}, s \in S, n \in \mathbb{N}_0\} \Big/ \begin{matrix} uss^{-1}v \sim uv \\ us^{-1}sv \sim uv \\ s \in S, u, v \text{ words} \end{matrix}$$

take the smallest equivalence relation satisfying these

<u>Example</u>: $S = \{a, b\}$: $ab^{-1}b^{-1}baaa^{-1}b$

<u>Lemma</u>: FS is a group under concatenation of words:

$$u \cdot v = uv.$$

$\square$

1 = empty word $\quad (x_1 \dots x_n)^{-1} = x_n^{-1} \dots x_1^{-1} \quad [(s^{-1})^{-1} =_{def} s]$

Call a word <u>simple</u> if no element s in it is succeeded or preceded by s^{-1}. Easy to observe: Every word is equivalent to a simple word.

<u>Example</u>: $a\underline{aa^{-1}}abb^{-1}a^{-1}abbbb^{-1}$

$\sim aa\underline{bb^{-1}}a^{-1}abbbb^{-1} \sim a\underline{aa^{-1}}abbbb^{-1} \sim aab\underline{bb^{-1}} \sim \underbrace{aab}_{a^2b}$

reasonable to write $\underbrace{aa\cdots a}_{n} \stackrel{\text{def}}{=} a^n$

$\underbrace{a^{-1}a^{-1}\cdots a^{-1}}_{n} \stackrel{\text{def}}{=} a^{-n}$

<u>Lemma</u>: Two different simple words are not equivalent.

Dummit-Foote: Algebra

Discussion on Tall: Algebra (groups)

$\boxed{\text{Let } X \text{ be the set of all simple words in } s, s^{-1}, s \in S.}$

Another concept: A group G acting on a set X: We have a group of permutations on X, called the symmetric group $\mathrm{Sym}(X)$.

bijections $X \xrightarrow{\simeq} X$

An action of G on X is a homomorphism $G \xrightarrow{\varphi} \mathrm{Sym}(X)$. We just write $gx = \varphi(g)(x)$

permutation

Claim: The free group $F(S)$ acts on the above set X.

Universal property of $F(S)$: For every group G and every map of sets $f: S \to G$ this determines a unique homomorphism of groups $F: FS \to G$ where $F(s) = f(s)$

thought of as a word with a single element s

(On a word, just take the product of $f(s)$ or $f(s)^{-1}$, s in the word. Equivalent words go to the same thing.)

To prove the claim, we need to construct a homomorphism

$$FS \to Sym(X)$$

By the universal property, all we need to do is construct a map of sets $S \to Sym(X)$.

The permutation corresponding to $s \in S$:

If w does not start with s^{-1}: $\boxed{w \longmapsto sw}$

If $w = s^{-1}u$: $\boxed{w \longmapsto u}$

It is obvious that this is a permutation. So FS acts on X

Let w be a simple word

$$[w] \cdot \underset{\underset{X}{\cap}}{\emptyset} = w.$$

By definition. These are different elements of X for different simple words w. So the classes $[w] \in FS$ are different.

$\square$

Generators and relations for groups.

$$\langle S \mid W \rangle$$

set

set of words in s, s^{-1}

(feel free to only use simple words).

means

$$F S \Big/ \text{smallest normal subgroup containing } W$$

more precisely: $\left([w] \text{ for } w \in W \right)$

Discussion:

Tuesday

Observation: Every group G can be easily expressed in the form $\langle S \mid W \rangle$ (generators and relations).

e.g. $S = G$ ← denote $g \in G$ as an element of S by (g)

$$W = \{ (g)(h)(gh)^{-1},\ g, h \in G \}$$

$$\langle S \mid W \rangle \xrightarrow{\cong} G$$

$$(g) \longmapsto g$$

normal subgroup

Homomorphism thm: $N \trianglelefteq G$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\text{projection}"'] & \Gamma \\ G/N \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

f homomorphism of groups

If $f(N) = \{e\}$

$\langle S \mid W \rangle \longrightarrow G.$

universal property characterizes G/N

The reason $\langle S \mid W \rangle \longrightarrow G$ is $\cong$ is because G satisfies the univ. property:

$$S \longrightarrow H \qquad G \longrightarrow H$$

$$w \in W \longmapsto \{e\}$$

$$\uparrow$$

$$(g)(h)(gh)^{-1}$$

The pushout of groups therefore suffices to be described in terms of generators and relations:

$$\begin{tikzcd}
\langle S \mid U \rangle \arrow[r, "f_1"] \arrow[d, "f_2"'] & \langle T_1 \mid V_1 \rangle \arrow[d] \\
\langle T_2 \mid V_2 \rangle \arrow[r] & \langle T_1 \sqcup T_2 \mid V_1 \cup V_2 \cup \{ f_1(s) f_2(s)^{-1} \mid s \in S \} \rangle
\end{tikzcd}$$

homomorphisms of groups

The pushout

This is the pushout (it satisfies the universal property).

$\therefore$ The pushout of groups <u>exists</u>.

Recall

<u>Theorem</u>: $U, V \subseteq X$ open, $U, V, U \cap V$ path-connected $x \in U \cap V$. Then we have a pushout:

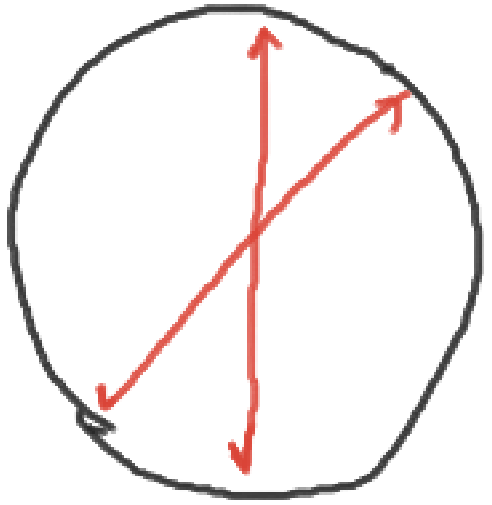
$$\begin{tikzcd}
\pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(U, x) \arrow[d] \\
\pi_1(V, x) \arrow[r] & \pi_1(X, x)
\end{tikzcd}$$

$\square$

This allows the calculation of $\pi_1(X,x)$ for any "reasonable" space X.

<u>Example</u>: $\mathbb{RP}^2 = D / z \sim -z$ if $|z| = 1$

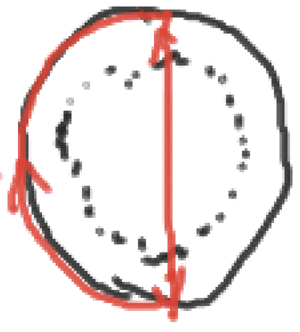
unit disk in $\mathbb{C}$



Calculate $\pi_1(\mathbb{RP}^2)$.

S^1

$U = \{ z \in D \mid |z| > \frac{1}{2} \}$ $\qquad V = \{ z \in D \mid |z| < 1 \} \cong *$. Choose $x \in U \cap V$



$S^1 \xrightarrow{\simeq} U$

$z \sim -z$

$\cong$ homotopy equivalence

S^1

$$\begin{array}{ccc} \mathbb{Z} & \pi_1(U \cap V, x) & \longrightarrow \pi_1(V, x) \\ \xi \;\downarrow & \langle \xi \mid \rangle \;\downarrow & \{e\} \\ \eta^2 \;\; \mathbb{Z} & \pi_1(U, x) \; \langle \eta \mid \rangle & \end{array}$$

$\xi \mapsto \emptyset$

$\xi \mapsto \eta^2$

$$
\begin{tikzcd}
\langle \xi \mid \ \rangle \arrow[r] \arrow[d] & \langle \ \mid \ \rangle \arrow[d] \\
\langle \eta \mid \ \rangle \arrow[r] & \langle \eta \mid \eta^2 \rangle
\end{tikzcd}
\qquad \mathbb{Z}/2
$$

$\therefore\ \pi_1 \mathbb{R}P^2 = \mathbb{Z}/2$

HW ⑤ Let X be the quotient of the unit disk $D \subset \mathbb{C}$ by $z \sim e^{2\pi i/n} z$, $z \in S^1$. Calculate $\pi_1(X)$.