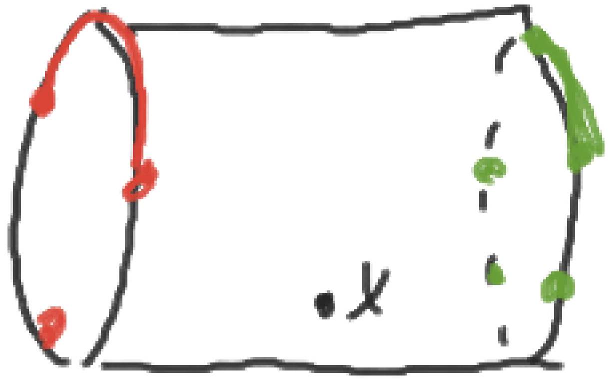
MATH 592 1/29/2024

Example: $X = S^1 \times [0,1] \Big/ \begin{array}{l}(z,0) \sim (e^{2\pi i/3} z, 0) \\ (z,1) \sim (e^{2\pi i/5} z, 1)\end{array}$ Find $\pi_1(X)$

$\subset \mathbb{C}$



$\simeq S^1/z \sim e^{2\pi i/3} z \simeq S^1$ $\qquad\qquad$ $\simeq S^1/z \sim e^{2\pi i/5} z \simeq S^1$

$U = S^1 \times [0,1) \big/ (z,0) \sim (e^{2\pi i/3} z, 0)$ $\qquad$ $V = S^1 \times (0,1] \big/ (z,1) \sim (e^{2\pi i/5} z, 1)$

$U \cap V = S^1 \times (0,1) \simeq S^1$, $x \in U \cap V$

$$\begin{array}{ccc} \pi_1(U \cap V, x) & \longrightarrow & \pi_1(U, x) \\ \downarrow & & \\ \pi_1(V, x) & & \end{array}$$

$$\begin{array}{ccc} \langle \xi \mid \ \rangle & \longrightarrow & \langle \alpha \mid \ \rangle \\ \downarrow & & \downarrow \\ \langle \beta \mid \ \rangle & \longrightarrow & \langle \alpha, \beta \mid \alpha^3 \beta^{-5} \rangle \end{array}$$

$\xi \mapsto \alpha^3$, $\xi \mapsto \beta^5$

Answer: The presentation for $\pi_1(X)$ is $\langle \alpha, \beta \mid \alpha^3 \beta^{-5} \rangle$

Warning: For two **finitely presented groups** (finitely many generators, relations), there **does not exist an algorithm** deciding whether the groups are isomorphic.

↑ we have proved! (Gödel)

HW 1 Calculate $\pi_1(X)$ where $X = S^1 \times [0,1]/\sim$ where $\sim$ is the smallest equivalence relation where $(z,0) \sim (z^7, 1)$.
(S^1 is the unit sphere in $\mathbb{C}$).

Example: $\bigvee_{Q} S^1 = S^1 \times Q \,/\, (1,t) \sim (1,t') \quad t,t' \in Q$

discrete set

$*$

loops indexed by Q.

$\pi_1\left(\bigvee_{Q} S^1, *\right) = FQ$

free group on Q

$U \cap V \simeq *$

The finite case: $\bigvee_{2=\{0,1\}} S^1$

$U = (S^1 \setminus \{-1\}) \vee S^1$ $\cong S^1$

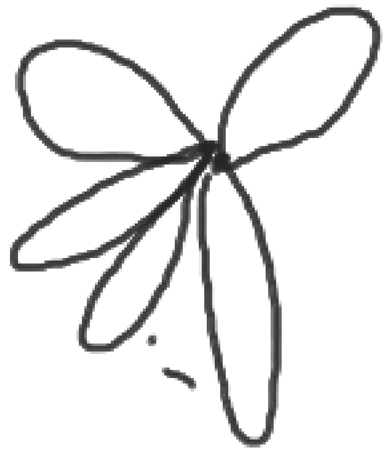
$V = S^1 \vee (S^1 \setminus \{-1\})$ $\cong S^1$

$\therefore$

$\pi_1(S^1 \vee S^1, *) = F\{0,1\} = \langle \alpha_0, \alpha_1 \mid \ \rangle$

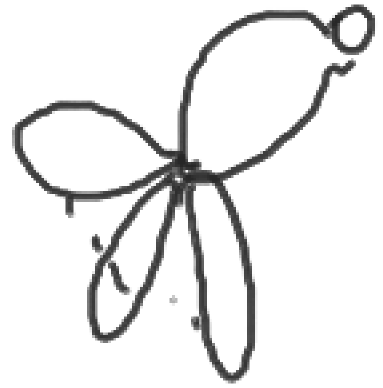
By induction, we can similarly prove

$$\pi_1\left(\bigvee_n S^1, *\right) = F\{0, \dots, n-1\}.$$

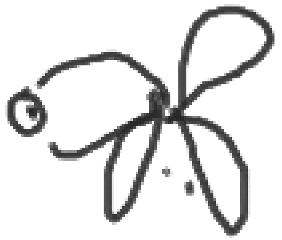


$$U \cap V \simeq \bigvee_{\{1, \dots, n-2\}} S^1$$

$$U = \bigvee_{n-1} S^1 \vee S^1 \setminus \{-1\}$$



$$V = S^1 \setminus \{-1\} \vee \bigvee_{\{1, \dots, n-1\}} S^1$$



$$\begin{tikzcd}
\langle x_1, \dots x_{n-2} \mid \; \rangle \arrow[r] \arrow[d] & \langle x_0, \dots x_{n-2} \mid \; \rangle \arrow[d] \\
\langle x_1, \dots x_n \mid \; \rangle \arrow[r] & \langle x_0, \dots, x_{n-1} \mid \; \rangle
\end{tikzcd}$$

The infinite case:

open set in $\bigvee_Q S^1$ is a subset whose intersection with every $\bigvee_F S^1$ for every $F \subset Q$ is open in $\bigvee_F S^1$.

$S^1 \to \bigvee_Q S^1$ continuous

The image is contained in some $\bigvee_F S^1$ $\quad F \subset Q$ finite

There holds a universal ~~property~~

$$\begin{tikzcd}
\underbrace{\pi_1\left(\bigvee_F S^1, *\right) \arrow[r] \quad \pi_1\left(\bigvee_{F'} S^1, *\right)}_{\text{all possible finite } F \subset Q} \arrow[rr, bend left] & & G \\
& \pi_1\left(\bigvee_Q S^1, *\right) \arrow[ur, dashed, "!"'] &
\end{tikzcd}$$

$$\pi_1\left(\bigvee_F S^1, *\right) \to \pi_1\left(\bigvee_{F'} S^1, *\right) \to \pi_1\left(\bigvee_Q S^1, *\right) \overset{!}{\dashrightarrow} G$$

A similar property holds for free groups.

This is an example of a colimit property in category theory.

Tomorrow's discussion

To summarise,

$$\pi_1\Big(\bigvee_Q S^1, *\Big) = FQ \quad (Q \text{ discrete set})$$

A graph X has a set of vertices V, and a set of edges E and two "incidence maps" $S, T : E \to V$.

The associated topological space

$$X = |X| = V \amalg E \times [0,1] \Big/ (e,0) \sim S(e),\ (e,1) \sim T(e) \text{ for } e \in E$$

the arrows don't matter.

$V = \{0, 1, 2, 3\}$

$E = \{a, b, c, d, e, f\}$

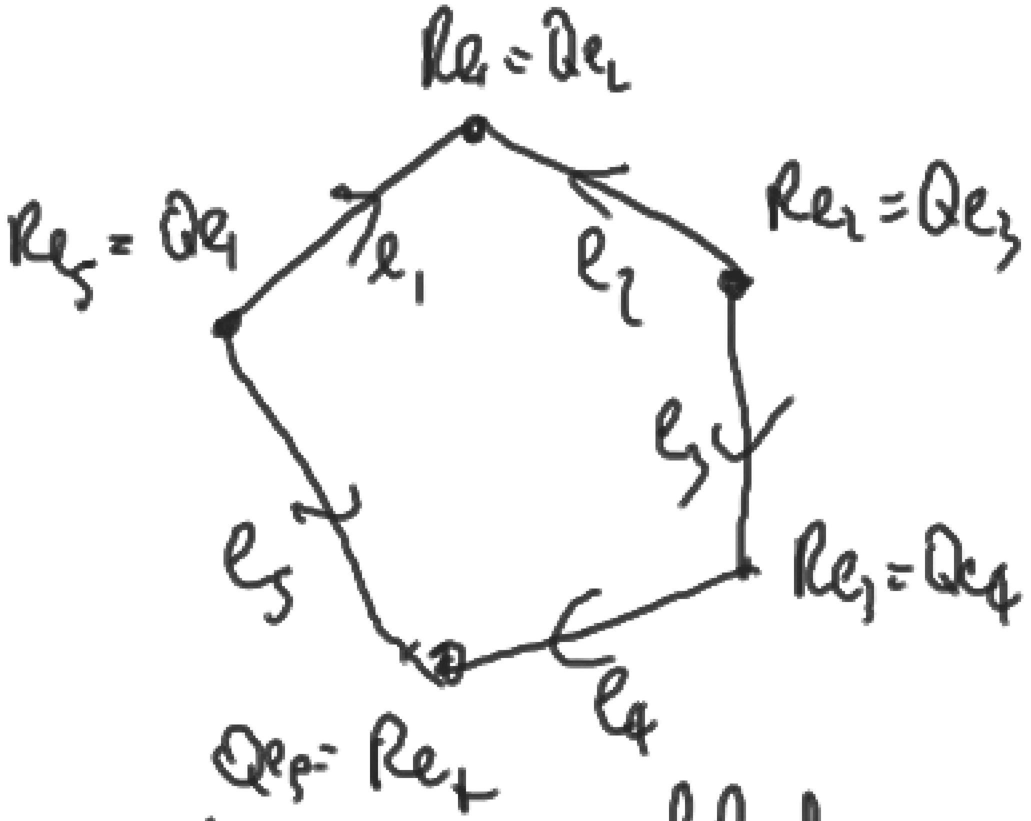
$$S(e) \xrightarrow{e} T(e)$$

A graph is <u>connected</u> when you can get from any vertex to any other vertex via edges. (direction doesn't matter)

(easily verify: equivalent to $|X|$ being a connected space).

$\Leftrightarrow$ path - connected

A *cycle* in a graph is a sequence of *different* edges $e_1, \dots, e_n$ $\quad n > 1$

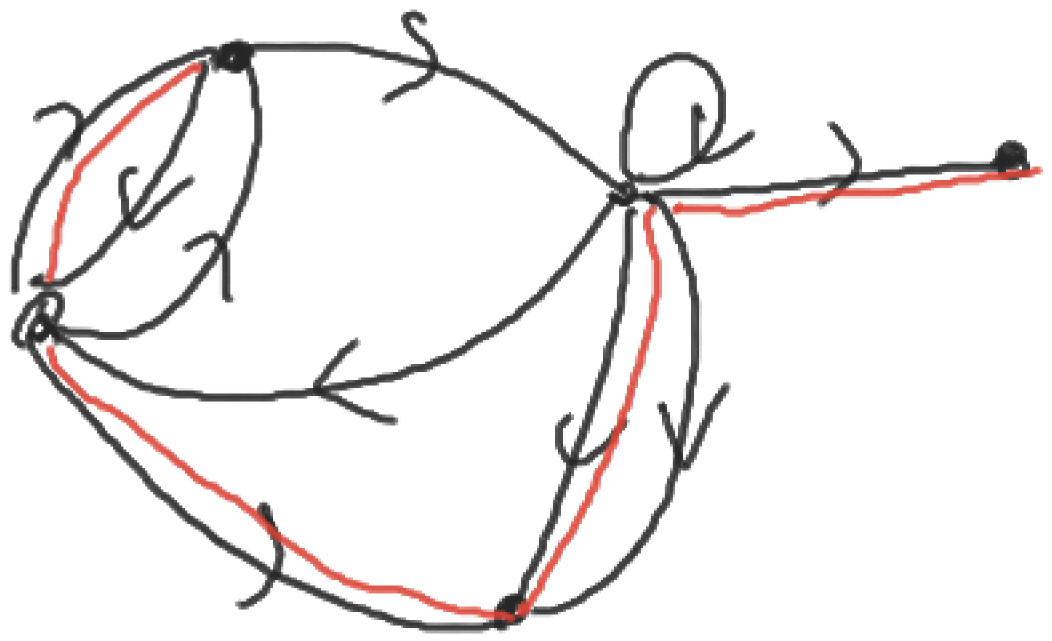


$$Qe_i = Re_{i-1} \qquad e_0 = e_n$$

$$Qe_i := Se_i \qquad Re_i := Te_i$$

or $Qe_i := Te_i$ and $Re_i := Se_i$.

A connected graph is called a *tree* if it has no cycles.



A maximal tree in a graph is called a *spanning tree*.

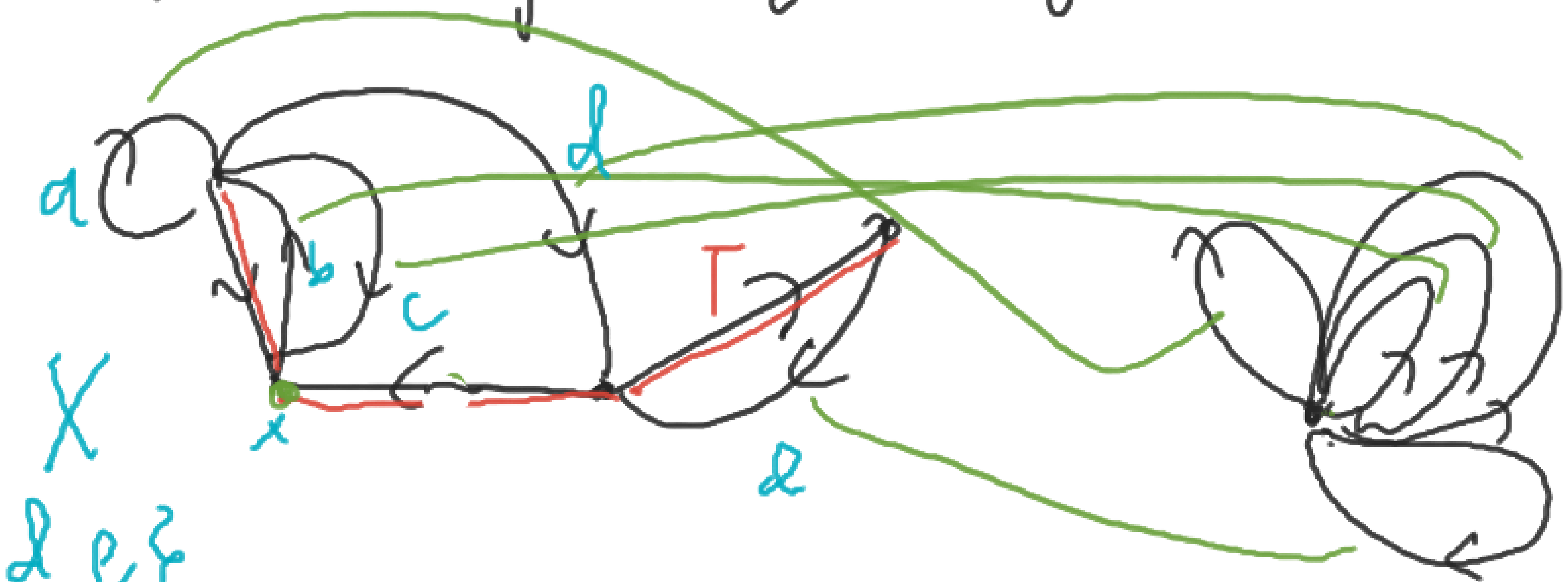
For a finite graph, all spanning trees have the same number of edges (matroid)

Theorem: A tree is contractible. Moreover, if X is a connected graph and $T \subseteq X$ is a spanning tree then the projection

$$|X| \longrightarrow |X|/|T| \simeq \bigvee_{Q} S^1$$

is a homotopy equivalence.

HW 2 Prove it when X has finitely many vertices and edges.



$$\pi_1(X,x) \cong F\{a,b,c,d,e\}.$$

Theorem: Let X be a connected graph and let T be a spanning tree. Choose a vertex x. Then

$$\pi_1(X, x) \cong FQ$$

where Q is the set of all edges of X which are not in the spanning tree T. $\square$