MATH 592 1/31/2024

Universal element Let $G : \mathcal{C} \to \mathcal{D}$ be a functor. Let $x \in \mathrm{Obj}\,\mathcal{D}$. A universal element for G, x is an object $y \in \mathrm{Obj}\,\mathcal{C}$ together with a morphism $x \xrightarrow{\gamma} G(y)$ which is universal in the following sense: Whenever we have a morphism $f : x \to G(z)$ for any $z \in \mathrm{Obj}\,\mathcal{C}$, there exists a *unique* morphism $f' : y \to z$ ($\in \mathrm{Mor}\,\mathcal{C}$) such that $f = G(f') \circ \gamma$:

$$\begin{tikzcd} x \arrow[r, "\gamma"] \arrow[dr, "f"'] & G(y) \arrow[d, dotted, "G(f')"] \\ & G(z) \end{tikzcd} \qquad \begin{tikzcd} y \arrow[d, dotted, "f'"] \\ z \end{tikzcd}$$

If a universal element y for a given functor $G : \mathcal{C} \to \mathcal{D}$ exists for every $x \in \mathrm{Obj}\,\mathcal{D}$, then one can prove that $y = F(x)$ is functorial in x. The functor F is called the <u>left adjoint</u> to G.

HW 3) Define $F(\varphi)$ for a morphism $\varphi : x \to x'$ and prove that F is a functor, where $F(x)$ is the universal element for G, x for every $x \in \mathrm{Obj}\,\mathcal{D}$. [apply the universality]

Example: $G : \mathrm{Grp} \to \mathrm{Set}$ be the forgetful functor. If S is a set, the universal element for G, S is the free group FS:

groups homomorphisms

sets maps

$$
\begin{tikzcd}
S \arrow[r] \arrow[dr, "f"'] & GFS \arrow[d, dotted, "Gf'"] \\
& G\Gamma
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
FS \arrow[d, dotted, "\exists!"', "f'"] \\
\Gamma
\end{tikzcd}
$$

FS considered as a set

$S \to \Gamma$ as sets

Γ group

f' as a map of sets

homomorphism of groups

The free group functor $F : Set \to Grp$ is left adjoint to the forgetful functor $G : Grp \to Set$.

Turning around arrows, we get a symmetrical notion of a co-universal element, when F is left adjoint to G then for $y \in \mathrm{Obj}\,\mathcal{C}$, $G(y)$ is co-universal w.r. to F, y, we call G right adjoint to F.

Colimits (dual notion - limits)

Let I be a small category. Let $\mathcal{C}$ be any category.

Diagram category $\mathcal{C}^I$:

Objects = functors $I \to \mathcal{C}$, Morphisms = Natural transformations

$\mathrm{const} = G : \mathcal{C} \to \mathcal{C}^I$ constant functor

$$x \in \mathrm{Obj}\,\mathcal{C} \longmapsto \begin{array}{ll} G(x)(i) = x & i \in \mathrm{Obj}\, I \\ G(x)(f) = \mathrm{Id}_x & f \in \mathrm{Mor}\, I. \end{array}$$

A universal element for G, $D : I \to \mathcal{C}$ $(x \in \mathrm{Obj}\,\mathcal{C})$, if one exists, is called the <u>colimit</u> of D.

<u>Example</u>: Pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ Z & \end{tikzcd}$$

$$\operatorname{colim}\begin{pmatrix} X \to Y \\ \downarrow \\ Z \end{pmatrix}$$

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] \arrow[dd] & & Y \arrow[dr] & \\ & \operatorname{colim} \arrow[rr] \arrow[dd] & & \operatorname{colim} \\ Z \arrow[dr] & & & \\ & \operatorname{colim} & & \end{tikzcd}$$

Universal.

The colimit argument for π_1.

In Top: A graph X, finite subgraphs $F \subseteq X$, $\subseteq$.

$$|X| = \operatorname*{colim}_{\substack{F \subseteq X \\ \text{finite}}} |F|$$

$$\pi_1(|X|, x) = \operatorname*{colim}_{\substack{F \subseteq X \\ \text{finite} \\ x \in F}} \pi_1(|F|, x).$$

<u>Last time</u>: X is a graph   set of vertices I_0

edges I_1

Suppose X connected, $x \in |I_0|$. Let T be a spanning tree of X

let E be the set of edges in I_1 not in T $(I_1 \setminus T)$.

$$\pi_1(X, x) \cong F(E).$$

<u>Example</u>:

X

$$\pi_1(X, x) = F\{a, b, c, d\} = F(a, b, c, d)$$

attach a 2-disk

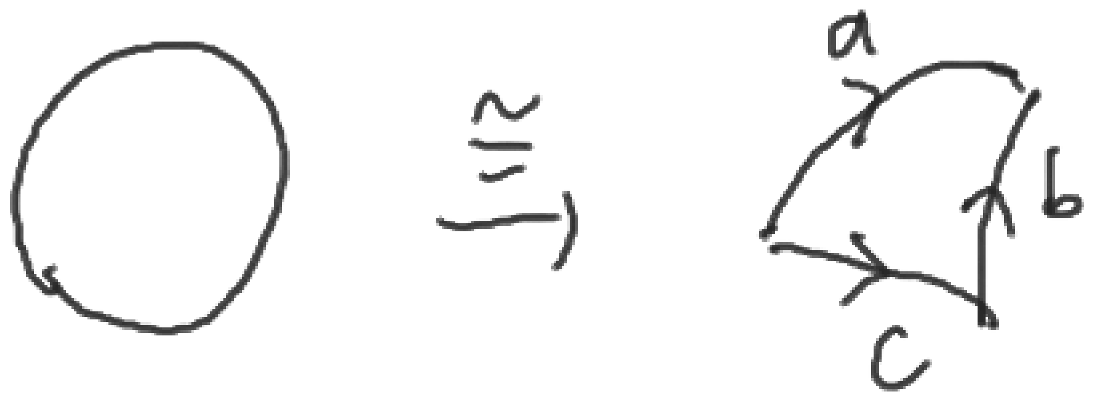
Say,

$D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

$\cup$

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$

Attaching map $S^1 \xrightarrow{\varphi} X$



$\simeq S^1$

$U \cap V \to U$

$\downarrow$

V

$t \mapsto ab^{-1}c^{-1}$

$\langle x \mid \ \rangle \to \langle a,b,c,d$

$\downarrow$

$\langle 1 \rangle$

Pushout: $\langle a,b,c,d \mid ab^{-1}c^{-1} \rangle$

I formed a space: $Y = \operatorname{colim} \begin{pmatrix} S^1 \xrightarrow{\varphi} X \\ \cap \\ D^2 \end{pmatrix}$

Alternatives: $b^{-1}c^{-1}a$, $c^{-1}ab^{-1}$, $ba^{-1}c$

$\pi_1(Y,x) = F(a,b,c,d) / ab^{-1}c^{-1}$

$\langle a,b,c,d \mid ab^{-1}c^{-1} \rangle$ ← smallest normal subgroup

A CW-complex X. Sets I_n of n-cells $n = 0, 1, 2, \ldots$ (some can be empty)

$$X = \operatorname{colim}\left(X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \right)$$

n-dimensional if $n = \max\{k \mid I_k \neq \emptyset\}$

such that: $X_{-1} = \emptyset$. For every $n = 0, 1, 2, \ldots,$ there is given a map (an attaching map) $\varphi_n : S^{n-1} \times I_n \longrightarrow X_{n-1}$

discrete

(in the example $I_2 = \{*\}$ $n = 2$)

and

$$X_n = \operatorname{colim} \left(\begin{array}{ccc} S^{n-1} \times I_n & \xrightarrow{\varphi_n} & X_{n-1} \\ \downarrow & & \\ D^n \times I_n & & \end{array} \right)$$

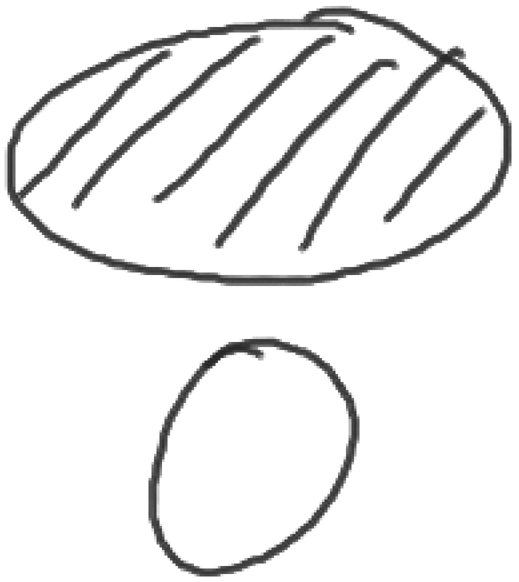
$D^0 = \{*\}$ $S^{-1} = \emptyset$

$D^n = \{(x_1, \ldots x_n) \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}$ $S^{n-1} = \{(x_1, \ldots x_n) \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$.

Example: S^n. One 0-cell $(1,0,0\dots 0)$

One n-cell

$D^n/S^{n-1} \cong S^n$



$X = \mathbb{R}$

$I_0 = \mathbb{Z} = X_0$ $\qquad I_1 = \mathbb{Z}$

different model: $D^1 = [0,1]$

$S^0 = \{0,1\}$

$\mathbb{Z} \times \{0,1\} \to \mathbb{Z}$

$(k,0) \mapsto k \qquad (k,1) \mapsto k+1$

1-dimensional CW-complex = graph

HW 4 Show a way of making each of the following spaces into a CW-complex:

a) $\mathbb{R}^n$

b) $Y = S^n \cup \{(t,0,\dots 0) \mid -1 \leq t \leq 1\}$ with subspace topology of $\mathbb{R}^{n+1}$

$\{(x_0, \dots x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$