MATH 592 2/2/2024

A CW-complex X is a space

$$X = \operatorname{colim}\left(X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \right)$$

X_n = n-skeleton

union with the union topology

We have $X_{-1} = \emptyset$ (if $X_{-1} = Z$, (X, Z) is a CW-pair)

variant

We have sets I_n (of n-cells) and attaching maps $\varphi_n : S^{n-1} \times I_n \to X_{n-1}$
and a pushout

$$\begin{tikzcd}
S^{n-1} \times I_n \arrow[r, "\varphi_n"] \arrow[d, hook] & X_{n-1} \arrow[d] \\
D^n \times I_n \arrow[r] & X_n
\end{tikzcd}$$

If X is a <u>connected</u> CW-complex (this happens if and only if X_1 is connected), if we choose $x \in X_0$, we can calculate $\pi_1(X,x)$ as <u>follows</u>:

a graph

Choose a spanning tree T of X_1 and let E be the set of 1-cells <u>not</u> in T ($I_1 \setminus T$). Then

$$\pi_1(X,x) \cong \langle E / R \rangle$$

The relations are obtained as follows. We have $\pi_1(X_1,x) \cong \langle E | \rangle$

For the 2-attaching map $\varphi_2 : S^1 \times I_2 \longrightarrow X_1$, if I choose $\alpha \in I_2$,

$$\begin{aligned} S^1 &\longrightarrow X_1 \\ t &\longmapsto \varphi_2(t,\alpha) \end{aligned}$$

determines a conjugacy class in $\pi_1(X_1,x)$.

Choose any representative w_α. Then $R = \{ w_\alpha \mid \alpha \in I_2 \}$

Note: cells of dimension $\geqslant 3$ do not change the fundamental group of a CW-complex.

Proof: Repeatedly apply the Seifert-van Kampen theorem to finite sub-CW complexes (meaning finitely many cells) and then use the colimit argument.

What happens one cell at a time:

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d] & Z \arrow[d] \\ D^n \arrow[r] & Y \end{tikzcd}$$

$n \geqslant 2$

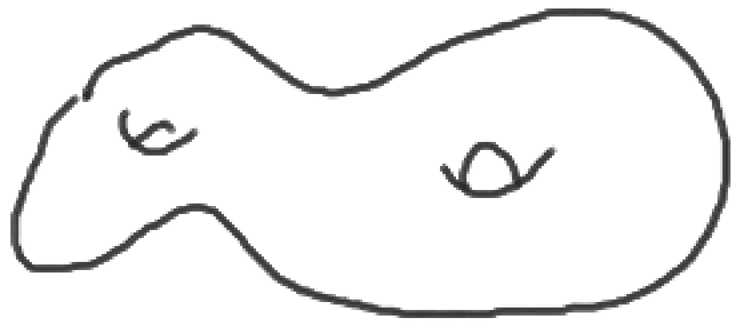
$$U = \operatorname{colim}\left(\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & Z \\ D^n \setminus \{(0,\dots,0)\} & \end{tikzcd} \right) \simeq Z$$

$V = D^n \setminus S^{n-1} \simeq *$

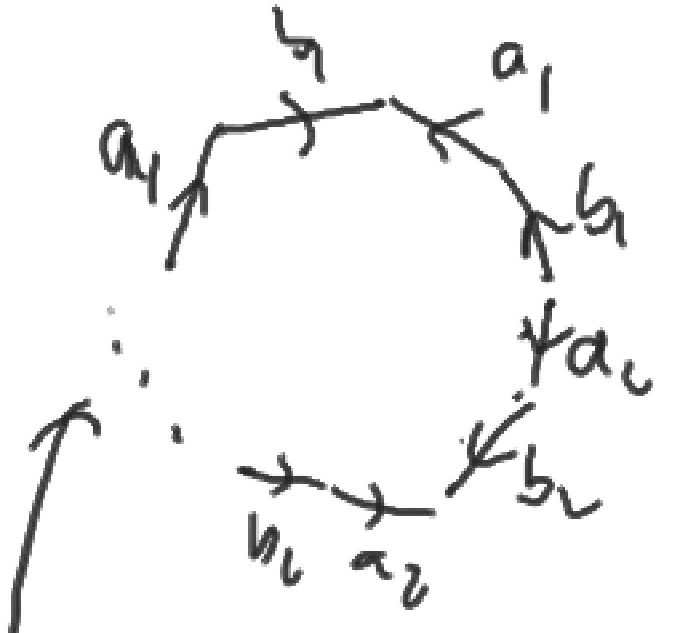
$U \cap V \simeq S^{n-1}$. If $n=2$, we add a relation to $\pi_1(Z)$.

If $n > 2$ $\quad \pi_1(V) = 0 \quad \pi_1(\underbrace{U \cap V}_{\simeq S^{n-1}}) = 0$. Attaching the cell does not change π_1. $\square$

<u>Examples</u>: Suppose we have a convex polygon (2-disk)



Suppose I identify edges of the polygon in pairs linearly homeomorphically. Then the quotient <u>always</u> is a compact topological surface (2-manifold).



repeat n times

$$\overbrace{T \# \cdots \# T}^{n \text{ times}} \longleftarrow$$ CW-complex X one 2-cell

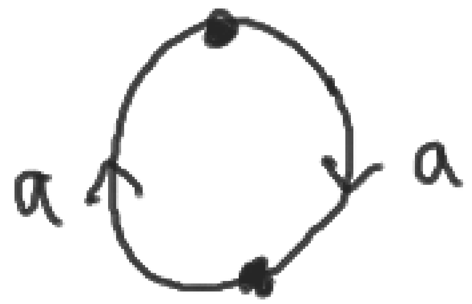
surface of a donut with n holes

X_1 - start with the cycle on the boundary, attach as prescribed. In this case, all the vertices are identified.

T

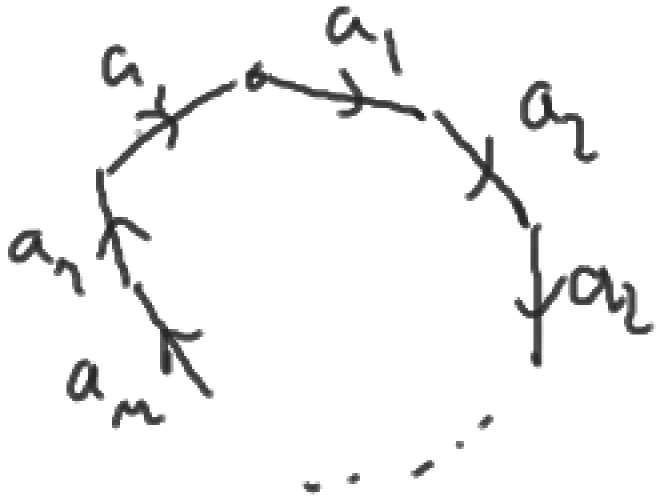
$T \# T$

$$\pi_1(\underbrace{T \# \cdots \# T}_{n}) = \langle a_1, b_1, a_2, b_2 \cdots a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle$$



Identify : We get $\mathbb{RP}^2$

(immersion into $\mathbb{R}^3$: Boy surface)



$\leftarrow \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n}$

$$\pi_1(\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2) = \langle a_1, \dots, a_n \mid a_1^2 \cdots a_n^2 \rangle$$

<u>Theorem</u>: Every compact topological surface is homeomorphic to precisely one

$$\underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n}, \quad n \geq 1$$

or

$$\underbrace{T \# \cdots \# T}_{n}, \quad n \geq 1$$

or S^2.

$\square$

One way to conclude that two groups in generators and relations are not isomorphic.

<u>Abelianization</u>: left adjoint to the inclusion functor $Ab \to Grp$

In generators and defining relations:

$$\langle S \mid R \rangle \rightsquigarrow \langle S \mid R \rangle_{ab}$$

as abelian groups (usually write additively, this time generators commute)

$$G \longmapsto G^{ab}$$

$G^{ab} = G /$ smallest normal subgroup containing all $\underbrace{ghg^{-1}h^{-1}}_{\text{commutators}}$, $g, h \in G$

$$\langle a_1, b_1, \dots a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle = \langle a_1, b_1, \dots a_n, b_n \mid \;\rangle_{ab} \cong \mathbb{Z}^{2n}$$

Classification of f.g. ab. groups

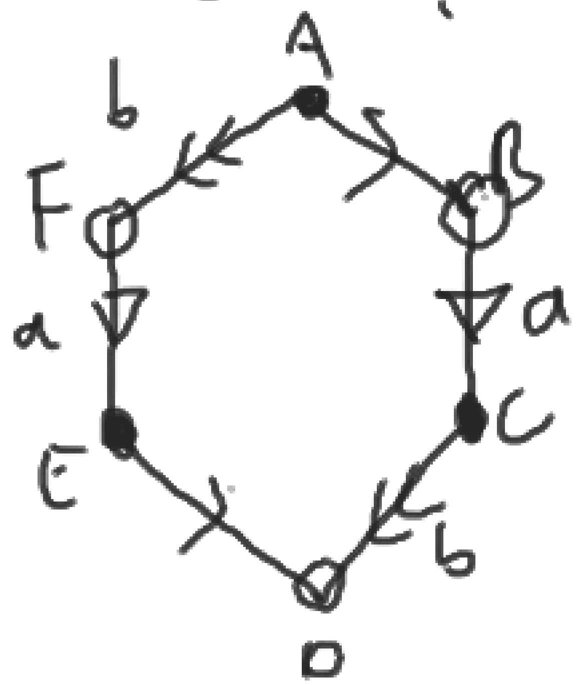
$$\langle a_1, \dots a_n \mid a_1^2 \dots a_n^2 \rangle \xmapsto{ab} \langle a_1, \dots a_n \mid 2a_1 + \dots 2a_n \rangle$$

$$\cong \mathbb{Z}/2 \oplus \mathbb{Z}^{n-1}$$

This distinguishes all the compact surfaces.

<u>Example</u>: What compact surface is obtained from a convex hexagon ABCDEF by identifying $AB \sim ED$, $BC \sim FE$, $CD \sim AF$ in this order?

X



$\boxed{XST}$

X_1 :

A
$= C$
$= E$

B
$= D$
$= F$

$\pi_1(X) = \langle a, b \mid aba^{-1}b^{-1} \rangle$

Spanning tree

choose a direction of each edge, follow it, ignore edges in spanning tree.

HW ⑤ In the polygon $A_1 A_2 \cdots A_{2n}$ $(A_i,\ i \in \mathbb{Z}/2n)$ identify $A_k A_{k+1}$ with $A_{k+n+1} A_{k+n}$ in this order, $k \in \mathbb{Z}/2n$. Compute the fundamental group and classify this surface.

⑥ (Not a surface!) In the convex octagon $ABCDEFGH$, identify linearly $AB \sim ED$, $BC \sim CD$, $EF \sim FG \sim GH \sim AH$ (in this order of vertices). Calculate the fundamental group in generators and relations.

Exam 4 Problems:

1 example of each type on Monday

More examples in Ben's discussion.

① Equivalence of categories

② CW-complex

③ Seifert-van Kampen directly ($S^1 \times [0,1]$ + something on ∂)

④ Polygon with attachments of edges, if it is a surface, recognize it