MATH 592

2/5/2024

Exam: 4 problems

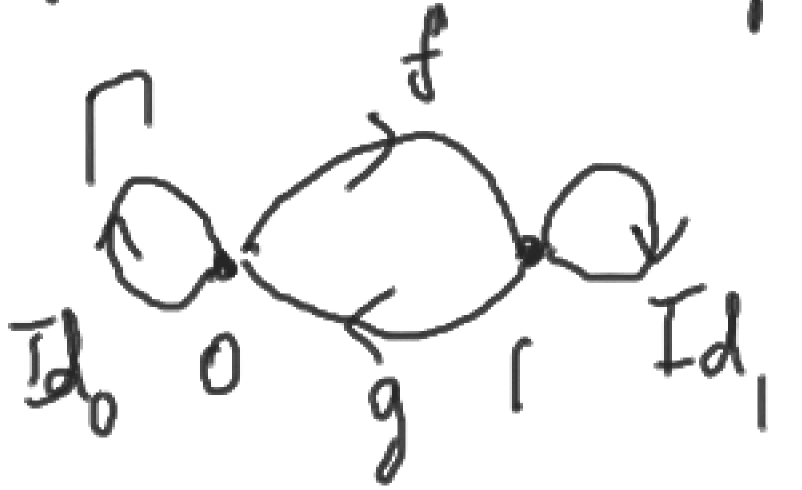
① Equivalence of category

Example: Groupoid Γ: $Obj(\Gamma) = \{0, 1\}$. There is precisely one morphism from any object to any other object.

Group G: One object, morphisms: $\mathbb{Z}/2$ (addition = composition).

Are G, Γ equivalent categories?

Solution:

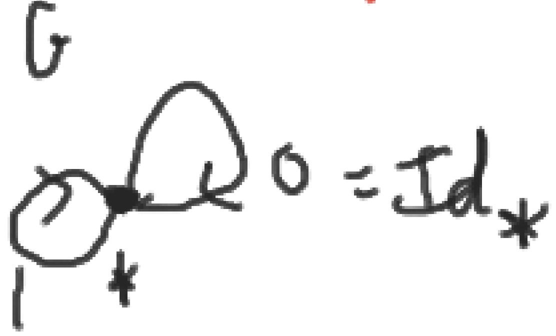


$f \circ g = Id_1$

$g \circ f = Id_0$

only one morphisms 1→1

0→0



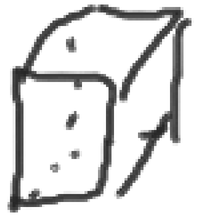
Skeleton of Γ : $\mathrm{Aut}_\Gamma(0) = \{e\}$ Not isomorphic groups,
(not unique)

Skeleton of G : $\mathrm{Aut}_G(*) = \mathbb{Z}/2$ ∴ Not equivalent categories.

<u>Problem 2</u>: Exhibit a CW-structure on the space X (to be given).

<u>Example</u>: $X = [0,1]^n$.

<u>Solution</u>: Strategy 1:



cells = faces of the cube. k-cells $= \{(t_1, \dots t_n) \mid t_i = 0 \text{ or } t_i = 1 \text{ when } i \notin S\}$, S given, $|S| = k$.

$I_k = \{(S,\varphi) \mid S \subseteq \{1, \dots n\}, |S| = k, \varphi : \{1,\dots n\}\setminus S \to \{0,1\}\}$ ← $2^{n-k}\binom{n}{k}$ k-cells

Attaching map of the k-cell corresponding to (S, φ) is:

$$\partial[0,1]^k \longrightarrow X_{k-1}$$

$S = \{i_1 < i_2 < \dots < i_k\}$

at least one coord $= 0$ or 1

$$(t_1, \dots t_k) \longmapsto (u_1, \dots u_n)$$

$u_i = \varphi(i)$ $\quad i \notin S$

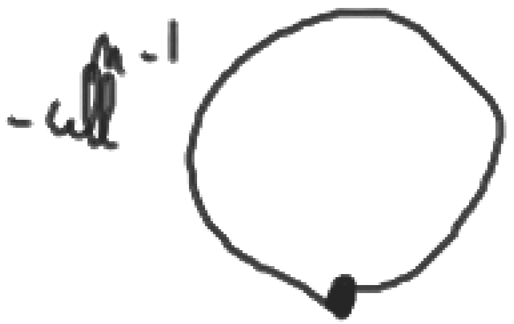
$u_{i_j} = t_j$

Model of $S^{k-1} \subset D^k$:

$$\partial[0,1]^k \subset [0,1]^k$$

By universality, get a continuous bijection $\underset{\text{compact}}{X} \to \underset{\text{Hausdorff}}{[0,1]^n}$ ∴ homeomorphism.

Strategy 2: $[0,1]^n \cong D^n$. We only need 3 cells:

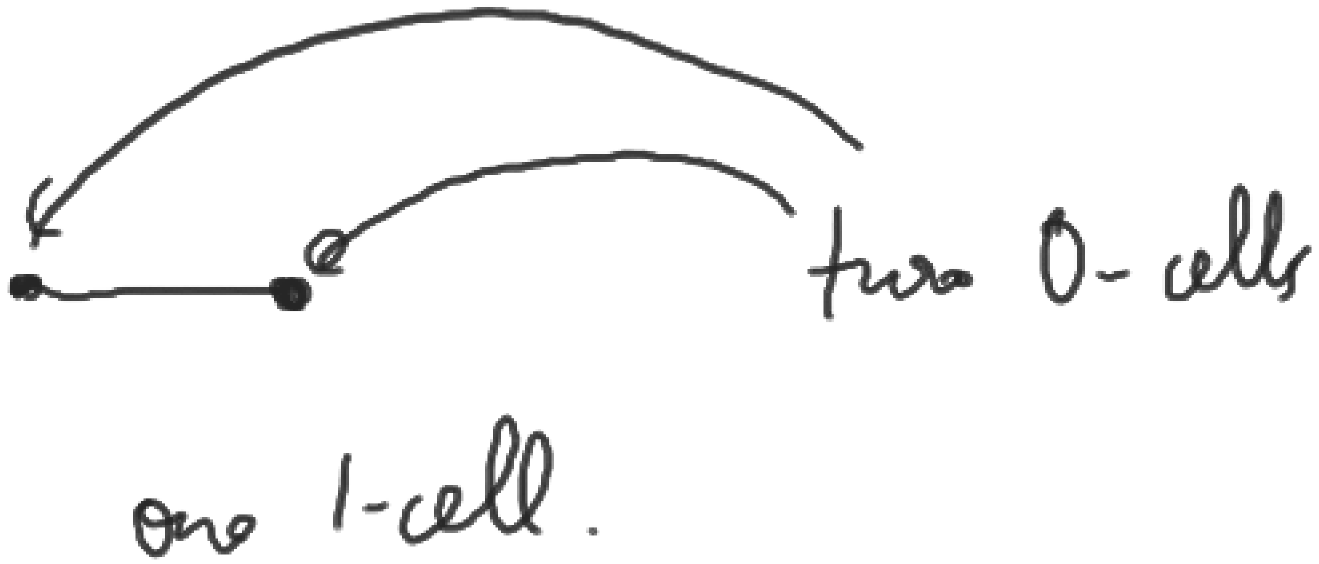


$D^{n-1}/S^{n-2} \cong S^{n-1}$

Attaching map $S^{n-2} \to *$ (unique such map)

Attaching map of the n-cell : $\mathrm{Id} : S^{n-1} \to S^{n-1}$

Special case for $n = 1$



③ Applying Seifert - Van Kampen theorem directly.

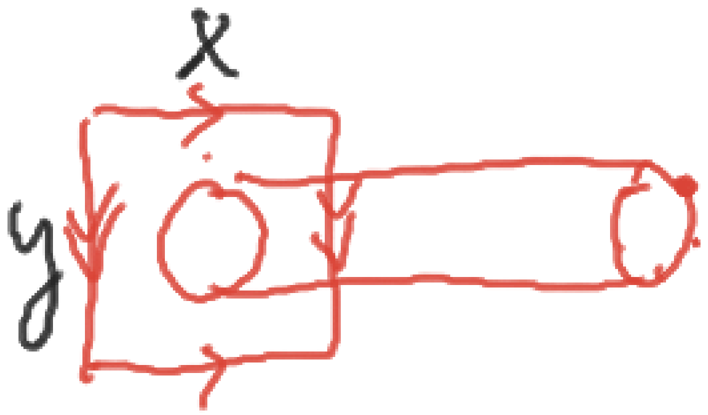
$$Z = \Big(\big(S^1 \times [0,1]\big) \sqcup \big([-2,2]\times[-2,2] \setminus \{(z,t)\in\mathbb{R}^2 \mid z^2+t^2<1\}\big)\Big) \Big/ \sim$$

unit plane in $\mathbb{C}$

$(z,1)\sim(ze^{2\pi i/9},1)$, $(x+iy,0)\sim(x,y)$, $(z,2)\sim(z,-2)$

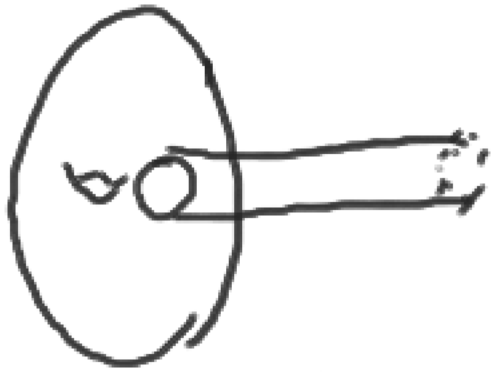
$(2,z)\sim(-2,z)$

Find $\pi_1(Z)$.



T-dish

U = Complement of $\{(z,1)\}$



U

$\simeq T \cdot$ open disk

$\simeq$ graph

V = complement of $T \cdot$ open disk

$/\sim \;\simeq S^1 \quad \left(S^1 /_{z \sim e^{2\pi i/q} z} \right)$

$\pi_1(S^1) = \mathbb{Z}$ generator: b

$U \cap V \simeq S^1$

generator a

$\langle a | \rangle \qquad a \mapsto b^q \qquad \langle b | \rangle$

$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r] \arrow[d] & \pi_1(V) \\ \pi_1(U) & \end{tikzcd}$$

$a \mapsto xyx^{-1}y^{-1}$

$\langle x, y | \rangle$

Answer: $\langle x, y, b \,|\, xyx^{-1}y^{-1}b^{-q} \rangle$

④ $\pi_1((\overset{\text{surface}}{\text{W}}\text{-complex})$

Identify the $\overset{\text{abelianised}}{\underset{\pi_1}{\text{surface}}} \leftarrow \mathbb{Z}^{2k}$

Example: $n = 4$

$\underbrace{T \# T \# \cdots \# T}_{k}$

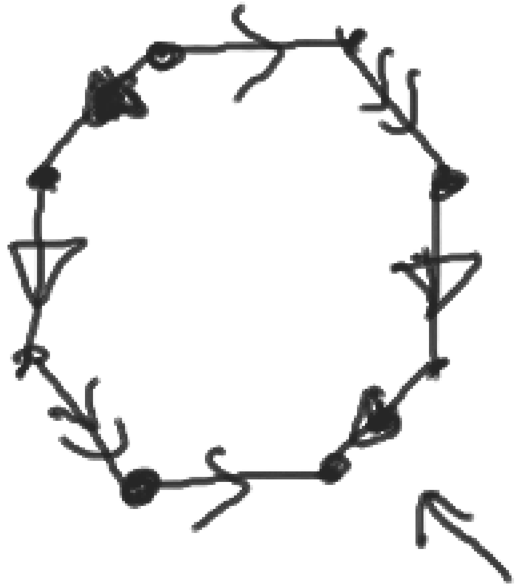
Id

$\mathbb{Z}^k \oplus \mathbb{Z}/2$

$\#_{k+1} \mathbb{RP}^2$

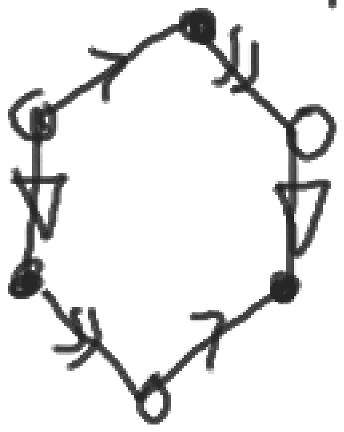
Solution of HW problem:

2n-gon



n even
$n = 2k$

$n = 3$



n odd

$n = 2k+1$

2 vertices

2k+1 edges

Spanning tree

$\langle a_1, \dots, a_{2k} \mid a_1 a_2 \cdots a_{2k} a_1^{-1} a_2^{-1} \cdots a_{2k}^{-1} \rangle$

2k edges

$\langle a_1, \dots, a_{2k} \mid a_1 a_2 \cdots a_{2k} a_1^{-1} a_2^{-1} \cdots a_{2k}^{-1} \rangle$

$\#_k T$

$n = 2k+1$ $n = 2k$

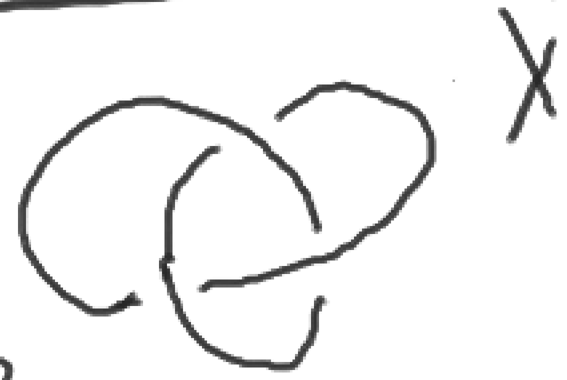
abelianised

$\langle a_1, \dots, a_{2k} \mid \rangle_{ab}$

$= \mathbb{Z}^{2k}$

Not on the exam

<u>Example</u>: $S^3 \supseteq$



is not a trivial knot.

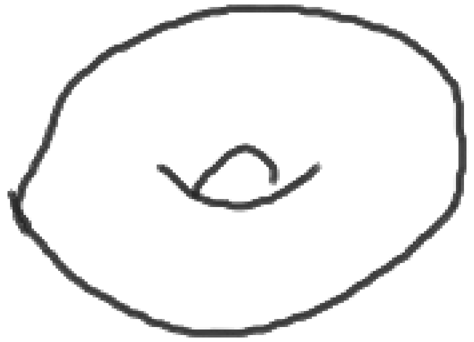
$$(S^3, X) \not\cong (S^3, S^1)$$

$$\nexists\, f : S^3 \xrightarrow{\cong} S^3$$

$$f : X \xrightarrow{\cong} S^1$$

X trefoil $\subseteq T = S^1 \times S^1$

$z \longmapsto (z^2, z^3)$



$D^4 = D^2 \times D^2$

$S^3 = \partial D^4$

Calculate $\pi_1(S^3 \setminus X)$

Thicken

$c \longmapsto a^3$

makes no diff

$$\begin{tikzcd}
\langle c \mid \ \rangle & & \langle a \mid \ \rangle \\
\pi_1(\underbrace{T \setminus X}_{\simeq S^1}) \arrow[rr] \arrow[d] & & \pi_1(\underbrace{(S^1 \times D^2) \setminus X}_{\simeq S^1}) \arrow[d] \\
\langle b \mid \ \rangle \quad \pi_1((D^2 \times \overset{S^1}{\widetilde{S^1}}) \setminus X) \arrow[rr] & & \pi_1(S^3 \setminus X) \cong \langle a, b \mid a^3 b^{-2} \rangle
\end{tikzcd}$$

$c \downarrow b^2$

$$\begin{tikzcd}
S^1 \times S^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S^1 \times D^2 \arrow[d] \\
D^2 \times S^1 \arrow[r] & S^3
\end{tikzcd}$$

Not abelian!

$\langle a, b \mid a^3 b^{-2} \rangle \to \Sigma_3$

group of permutations on $\{1,2,3\}$

$a \mapsto (123) \qquad b \mapsto (12)$

$\pi_1(S^3 \setminus S^1) \cong \mathbb{Z}$

not isomorphic!

Discussion: Tomorrow EH 5281 1-2 PM

more exam-type problems!