MATH 592 2/9/2024

Covering spaces

$\varphi: \tilde{X} \longrightarrow X$ continuous

(covering space of X)

$X = \bigcup U_i$ open covering, $\varphi^{-1}(U_i) = \coprod_{j \in I_i} U_{ij}$

$\varphi: U_{ij} \xrightarrow{\cong} U_i$

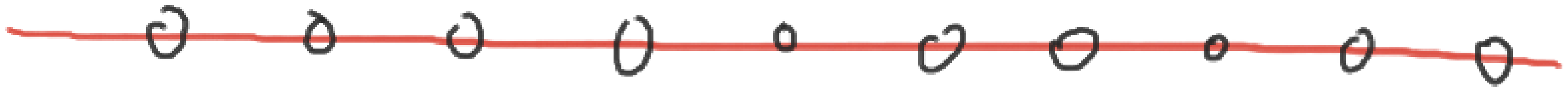
homeomorphism

U_i = fundamental neighborhoods

Example: $\varphi : \mathbb{R} \longrightarrow S^1$
$t \longmapsto e^{2\pi i t}$

$$U_n = \left(\frac{n}{2} - \frac{1}{2}, \frac{n}{2} + \frac{1}{2}\right), \quad n \in \mathbb{Z}$$

(any open interval of length ≤ 1).

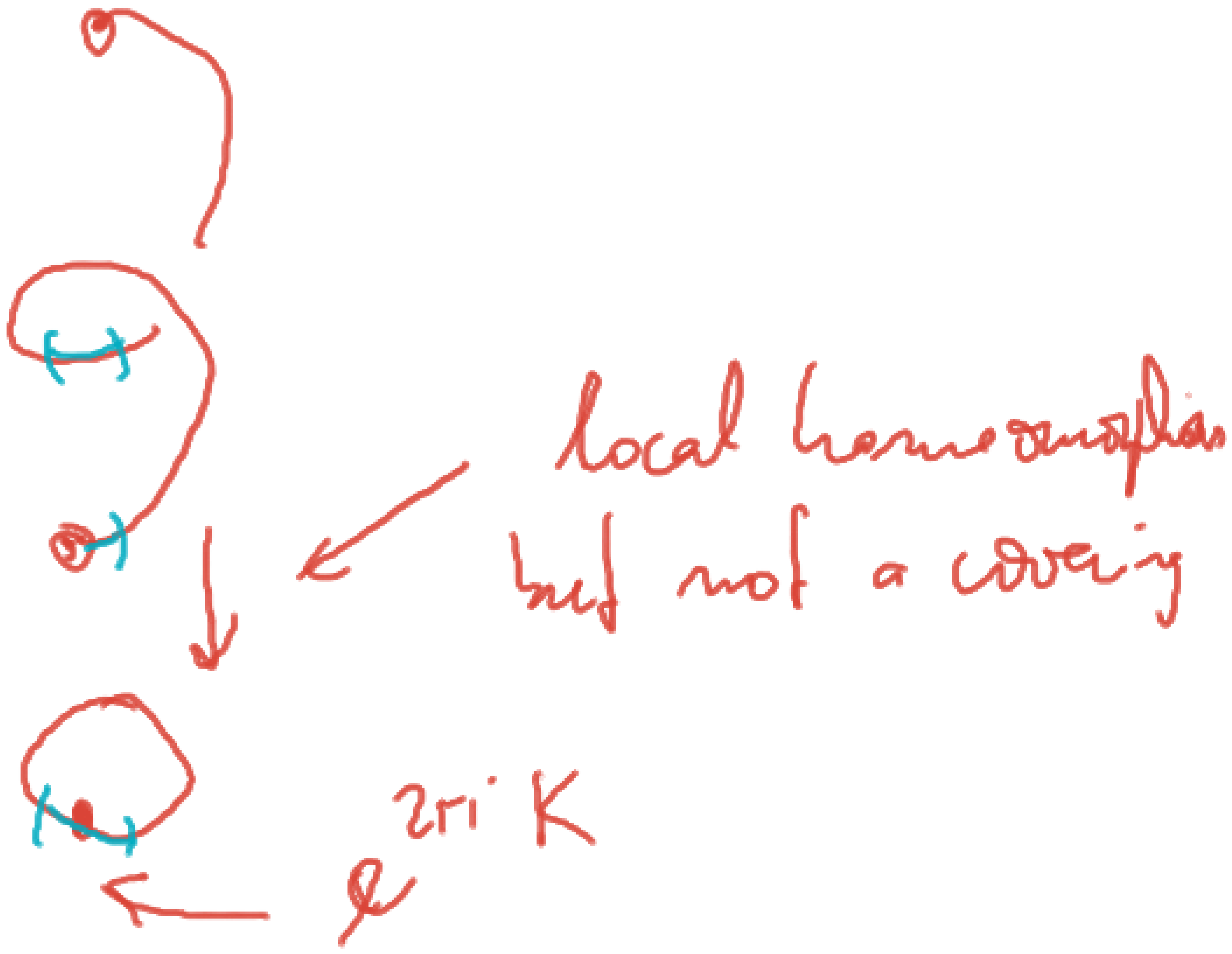


(This is how we calculated $\pi_1(S^1)$.)

Non-example:

$$(-K, L) \xrightarrow{t \mapsto e^{2\pi i t}} S^1$$

$-\infty < K < L < \infty$



Why is a covering helpful? Recall lifting of

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ {[0,1]} \arrow[r] \arrow[ur, dotted] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ {[0,1] \times [0,1]} \arrow[r] \arrow[ur, dotted] & S^1 \end{tikzcd}$$

unique subject to lifting one point.

Lifting property of covering spaces:

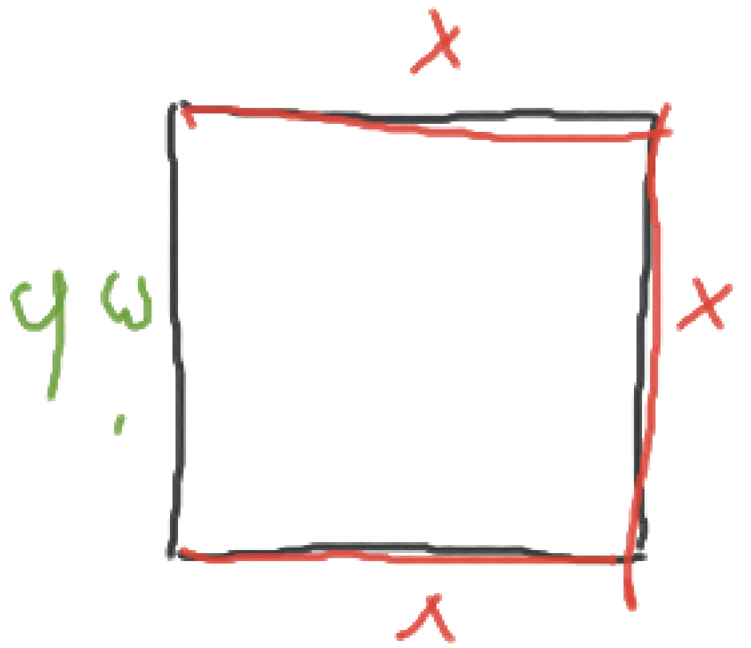
Lemma: Let $D = [0,1]^n$.
Let $\varphi : \tilde{X} \to X$ be a covering. Any diagram as below can be completed in Top:

$$\begin{tikzcd}
D \arrow[r, "\tilde{f}"] \arrow[d, "x \mapsto (x,0)"'] & \tilde{X} \arrow[d, "\varphi"] \\
D \times [0,1] \arrow[r, "f"] \arrow[ur, dashed, "F"] & X
\end{tikzcd}$$

Proof: $D = [0,1]^n$. $\exists N$ $f\Big(\underbrace{\left[\frac{k_0}{N}, \frac{k_0+1}{N}\right] \times \cdots \times \left[\frac{k_n}{N}, \frac{k_n+1}{N}\right]}_{Q_{k_0,\dots,k_n}}\Big) \subseteq U_i$

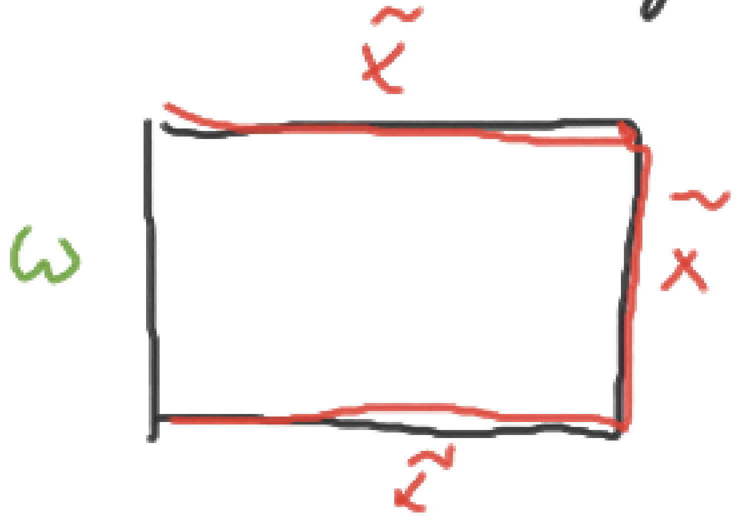
We can arrange these lexicographically, the tuples
$(k_0, \dots, k_n)$
so the intersection of $Q_{k_0, \dots k_n}$ with $\bigcup_{(l_0, \dots l_n) < (k_0, \dots k_n)} Q_{l_0, \dots l_n}$ is
connected. $\square$

<u>Lemma</u>: Let $\varphi : \tilde{X} \to X$ be a covering. $\varphi(\tilde{x}) = x$.
Then $\pi_1(\varphi) : \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$ is injective.
Proof: $\omega : [0,1] \to \tilde{X}$, $\omega(0) = \omega(1) = \tilde{x}$. Suppose it is based-homotopic
to a constant loop after composing with φ: $h_0(z) = \omega(z)$
$h : [0,1] \times [0,1] \to X$ $\quad h_t(0) = h_t(1) = h_1(z) = x$



$\subseteq X$

Just lift this using the lemma where we lift $\varphi\omega$ to ω:



$\therefore [\omega] = 0 \in \pi_1(\tilde{X}, \tilde{x}).$ $\square$

HW: Let G-set be the category of sets with G-action and equivariant maps.

suppress from no.

$$G \xrightarrow{\ \checkmark\ } \mathrm{Sym}(S)$$

homomorphism

all permutations on S.

action

$f : X \to Y$

$f(gx) = gf(x)$

1. Prove that every G-set X is isomorphic to a G-set of the form

$$\coprod_{i \in I} G/H_i$$

orbits

$H_i \subseteq G$

$G/H = \{ gH \mid g \in G \}$

$g'(gH) = g'gH.$

(2) Describe all the morphisms in G-Set

$$G/H \longrightarrow G/K \quad , \; H, K \text{ subgroups of } G.$$