MATH 592 2/12/2024

Homotopy lifting property

$$\begin{tikzcd}
Y \arrow[r, "\tilde{f}"] \arrow[d, "0"'] & \tilde{X} \arrow[d, "\varphi"] \\
Y \times [0,1] \arrow[r, "F"] \arrow[ur, dotted] & X
\end{tikzcd}$$

| | |
|---|---|
| $\forall n$ $\quad Y = D^n$ | $\varphi$ Serre fibration |
| $\forall Y$ | $\varphi$ Hurewicz fibration |

Hatcher

tom Dieck

**Theorem**: Every covering is a Hurewicz fibration.

**Proof**: For each $y \in Y$, we can lift the path $\{y\} \times [0,1] \xrightarrow[F]{} X$. (This implies uniqueness, not a part of this statement)

Need to prove the lifting $Y \times [0,1] \to \tilde{X}$ is continuous.

$\forall y \in Y \quad \forall t \in [0,1] \quad \exists$ neighborhood $W_t$ of $t$ and $V_t$ of $y$

fix $y$ such that $f(V_t \times W_t) \subseteq$ fundamental neighborhood $U_i$ in $X$.

Because $[0,1]$ is compact, only need finitely many $V_{t_j}$ to cover $[0,1]$,

Hence, $W_t$ can be chosen uniformly $W = \bigcap W_{t_j}$.

$W \times [0,1]$

$0 = t_0 < t_1 < \cdots < t_N = 1$

in each $f([t_i, t_{i+1}], W \times [t_i, t_{i+1}])$

$\subseteq$ fundamental neighb. $U_i$

lift is continuous on

$W \times [t_i, t_{i+1}]$

(it is the "obvious" lift, since the lift is unique).

since the gluing.

$$\begin{array}{ccc} \varphi^{-1}(U_i) & \longrightarrow & U_i \\ \text{\rotatebox{90}{$\cong$}} & \nearrow_{\text{proj}} & \\ U_i \times S_i & & \end{array}$$

$S_i$ discrete

Gluing $\therefore$ continuous on $W \times [0,1]$. $\square$

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Usual assumption on $X$: path-connected $\underbrace{\text{locally path-connected}}_{\substack{\forall x\ \forall U \ni x \text{ open} \\ \exists \text{ neighborhood of } x \\ \text{which is path-connected.}}}$

PCLPC

<u>Theorem</u>: If $\varphi : \tilde{X} \to X$ is a covering, $x \in X$, $\tilde{x} \in \tilde{X}$, $\varphi(\tilde{x}) = x$,

Suppose $Y$ is PCLPC. Given $f : Y \to X$, $f(y) = x$. Then there exists a lift $\tilde{f} : Y \to \tilde{X}$, $\tilde{f}(y) = \tilde{x}$, $\varphi \circ \tilde{f} = \varphi$ if and only if

$\pi_1(f)\left(\pi_1(Y,y)\right) \subseteq \pi_1(\tilde{X}, \hat{x}))$

$$\boxed{\text{Recall that } \pi_1(\tilde{X},\tilde{x}) \xrightarrow[\subseteq]{\pi_1\varphi} \pi_1(X,x)}$$

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}) \arrow[d, "\varphi"] \\
(Y, y) \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & (X, x)
\end{tikzcd}
\qquad \text{iff} \qquad
\begin{tikzcd}
 & \pi_1(\tilde{X}, \tilde{x}) \arrow[d, "\pi_1 \varphi"] \\
\pi_1(Y, y) \arrow[r] \arrow[ur, dashed] \arrow[ur, phantom, "\circledast"'] & \pi_1(X, x)
\end{tikzcd} \subseteq$$

Necessity obvious (functoriality).

<u>Proof</u>: $Y$ path connected, $z \in Y$ choose a path from $y$ to $z$ and lift it. Consistency $\Leftarrow \circledast$.


Continuity $\Leftarrow$ $Y$ LPC. $\square$

<u>Comment</u>: A covering is a local homeomorphism $\Rightarrow$ open.

(image of an open set is open).

($\therefore$ A bijective covering is a homeomorphism.)

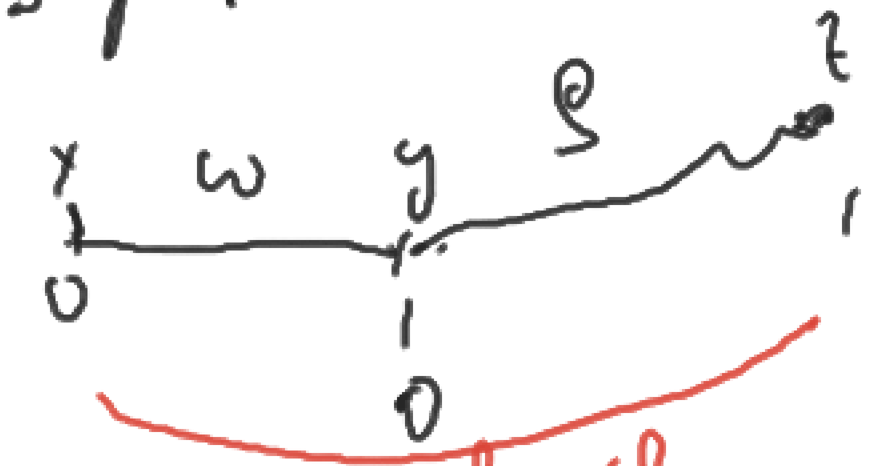
Inadequate text: Singer - Thorpe: ...

A connected covering of an PCLPC space is path-connected.

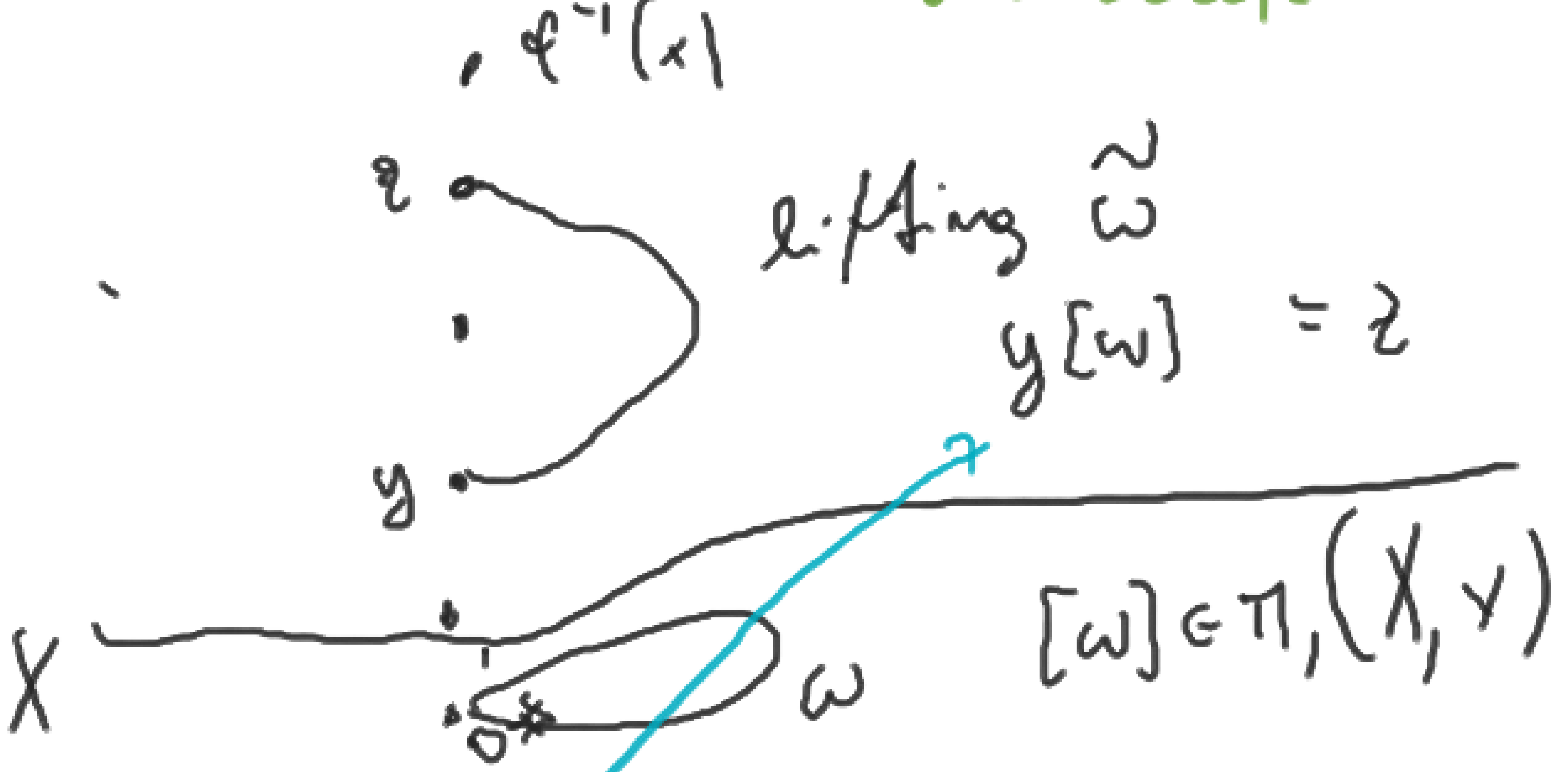
HW (1) Prove that if $X$ is PCLPC and $\pi_1(X)=0$ then every connected (non-empty $\tilde{X}$) covering $\varphi:\tilde{X}\to X$ is a homeomorphism.

Idea for classifying coverings: let $x \in X$, let $\varphi : \tilde{X} \to X$ be a covering. Then $G = \pi_1(X,x)$ acts on $\varphi^{-1}(x)$ ← a discrete set by lifting paths.

<u>Caution</u>:




We denote the composite path by $\omega\xi$.




This definition gives a <u>right action</u> of $G$ on $S = \varphi^{-1}(x)$.

$g \in G,\ s \in S \longmapsto sg$

For every group $G$, we have the <u>opposite group</u> $G^{op}$

$$g \cdot_{G^{op}} h := h \cdot_G g.$$

A right action of $G$ is a (left) action of $G^{op}$.

But in fact, for a group $G$, there is an isomorphism

$$\begin{aligned} G &\longrightarrow G^{op} \\ g &\longmapsto g^{-1}. \end{aligned}$$

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So $\pi_1(X, x)$ acts on $\varphi^{-1}(x)$ by a left action where $[\omega]$ acts by lifting $\overline{\omega}$ $\quad ([\overline{\omega}] = [\omega]^{-1})$.

Suppose we choose an $x \in X$. We have two categories of coverings of $X$.

① $\mathrm{Cov}(X)$ : Objects : coverings $\varphi : \tilde{X} \longrightarrow X$. Morphisms : $\varphi \to (\psi : \bar{X} \to X)$

commutative diagrams

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & \bar{X} \arrow[dl, "\psi"] \\ & X & \end{tikzcd} \qquad (\dagger)$$

HW ② Prove that for $f$ as above in $(\dagger)$, $f$ is also a covering.

We constructed a functor

$$\mathrm{Fib}_x : \mathrm{Cov}(X) \longrightarrow \pi_1(X,x)\text{-Set}$$
$$(\varphi : \tilde{X} \to X) \longmapsto \varphi^{-1}(x).$$

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Theorem (to be proved): If $X$ is a connected CW-complex then $\mathrm{Fib}_x$ is an equivalence of categories.

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This generalises to $X$ PCLPC, SLSC iff semilocally simply connected

$\forall x \in X$ $\exists U \ni x$ open $\exists$ neighborhood $V$ of $x$, $V \subseteq U$, $\pi_1(V,x) \xrightarrow{0} \pi_1(X,x)$.

**Examples:** Hawaiian earring  $= H$

$$\bigcup_{n\in\mathbb{N}} \frac{1}{n}\left(S^1 + \frac{1}{2}\right) \subset \mathbb{R}^2$$

(subspace) not SLSC

$\Big(\nexists$ $\underbrace{\text{connected covering with } \pi_1 = 0}_{\text{universal covering}}\Big)$

Cone on Hawaiian earring

$$\bar{H} := H \times [0,1] \,/\, H \times \{1\}.$$

$H \times \{0\} \subseteq \bar{H}$ but $\bar{H}$ is contractible.

HW 3 Let $G$ be a (discrete) group.
Let $H \subseteq G$ be a subgroup. Prove:

$$\mathrm{Aut}_{G\text{-Set}}(G/H) \cong W(H)$$

Group of automorphisms.

$N(H) = \{g \in G \mid g^{-1}Hg = H\}$

$H \lhd N(H)$.

$W(H) := N(H)/H$.

discussion:
Group actions, categories
G-Set

4 Let $G = \langle a, b \mid bab^{-1}a^{2} \rangle$. Find a subgroup $H \subset G$ and a morphism $G/H \to G/H$ in G-Set which is not an isomorphism.