MATH 592 2/14/2024

last time: X connected CW-complex, $x \in X$

Cov_X = category of covering spaces of X, $\underbrace{\text{deck transformations}}$

$$\begin{tikzcd} \tilde{X} \arrow[rr] \arrow[dr, "p"'] & & \bar{X} \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

$\mathrm{Fib}_x : \mathrm{Cov}_X \longrightarrow \pi_1(X,x)\text{-Set}$

$(p: \tilde{X} \to X) \longmapsto p^{-1}(x).$

Variant: $\mathrm{Cov}_{(X,x)}$ ← based coverings

objects: $\tilde{X}$ connected $\begin{array}{c} \tilde{X} \\ \downarrow p \\ X \end{array}$ covering, $\begin{array}{c} \tilde{x} \in \tilde{X} \\ \downarrow p \\ x \end{array}$

morphisms

$$\begin{tikzcd} \tilde{x} \arrow[rr, bend left, mapsto] \arrow[dr, mapsto, bend right] & \tilde{X} \arrow[r] \arrow[d, "p"'] & \bar{X} \arrow[dl, "q"] & \bar{x} \arrow[dll, mapsto, bend left] \\ & X & & \\ & x & & \end{tikzcd}$$

<u>Theorem</u>: The fiber functor

$$\mathrm{Fib}_x : \mathrm{Cov}_{(X,x)} \longrightarrow \left(\text{Subgroups of } \pi_1(X,x),\ \subseteq\ \right)$$

$$\begin{array}{ccc} \tilde{x} \in \tilde{X} & & \\ \downarrow \quad \downarrow & \longmapsto & \pi_1(\tilde{X},\tilde{x}) \\ x \in X & & \text{Image} \left\{ \subseteq \ \big\downarrow \pi_1(p) \right. \\ & & \pi_1(X,x) \end{array}$$

A partially ordered set (POSET) always makes a category in this fashion.

if $H \subseteq K$ there is precisely one morphism $H \to K$
if not, there is no morphism $H \to K$.

The lifting theorems immediately give that these functors are fully faithful

injection on morphisms between any two objects

onto on morphisms between any two objects.

The key piece which completes the proof of these theorems:

For every $H \subseteq \pi_1(X,x)$ $\exists$ a connected covering space $p: \tilde{X} \to X$, $\tilde{x} \mapsto x$

such that $\pi_1(p)(\pi_1(\tilde{X},\tilde{x})) = H \subseteq \pi_1(X,x)$. We can deduce this if we have it for $H = \{e\}$. In this case, $p: \tilde{X} \to X$ is called the universal covering. (A path-connected space with trivial fundamental group is called simply connected.)

Our plan: Very hands-on about coverings of (connected) CW-complexes

Theorem: For a covering $p: \tilde{X} \to X$ of a CW-complex X, $\tilde{X}$ is also a CW-complex.

Brief intermezzo: categorical limits (dual to colimits)

turn around all arrows.

$$L := \lim \left(\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \right) = \text{universal element of this diagram.}$$

A limit of this diagram is called a pullback.

$$\begin{tikzcd} L \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & L \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd}$$

Note: In Set, Top, the pullback is constructed as follows:

If $Z = *$, $\lim \left(\begin{matrix} X \\ \\ Y \end{matrix} \right)$ (is called a **product**)

is the Cartesian product.

$$\lim \left(\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \right) = X \times_Z Y = \{ (x,y) \in X \times Y \mid f(x) = g(y) \}$$

(in Top, subspace topology)

If we have a pullback diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "\bar{f}"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ Y \arrow[r] & Z \end{tikzcd}$$

if f is a covering, so is $\bar{f}$.

Proof of Theorem ⊛: A covering of D^n is a $\coprod_I D^n$.
(because a connected covering is a homeomorphism).

This gives a recipe, for a cover $p : \tilde{X} \longrightarrow X$ where X is a CW-complex, to prove by induction that for the covering $p_n : p^{-1}(X_n) \to X_n$ $p^{-1}(X_n)$ is a CW-cx. of dim $\leq n$. Assuming the induction hypothesis for $n-1$, take all the closed n-cells, pulling back p to them gives a disjoint union of n-disks and an attaching map $\coprod_J S^{n-1} \longrightarrow p^{-1}(X_{n-1})$

$\coprod_J D^n$

$$\begin{tikzcd}
\coprod_J S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & p^{-1}(X_{n-1}) \arrow[d] \\
\coprod_J D^n \arrow[r] & \tilde{X}_n
\end{tikzcd},$$

get a map $\varphi_n : \tilde{X}_n \longrightarrow p^{-1}(X_n)$

which is continuous and bijective. We show that φ_n is closed.

A set in a CW-cx is closed iff its pullback to every closed cell D^k is closed in D^k. One verifies the same is true for $j^{-1}(X_n)$.

(every cell is covered by finitely many fundamental neighborhoods).

Repeat this argument again when taking colim. $\square$

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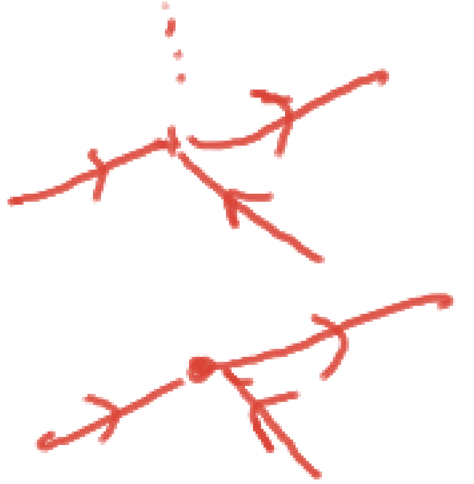
① Covering of graphs. Let Γ be a connected graph.

Its fundamental group $\pi_1(\Gamma, x)$ is <u>free</u> on edges not in a given spanning tree.

vertex

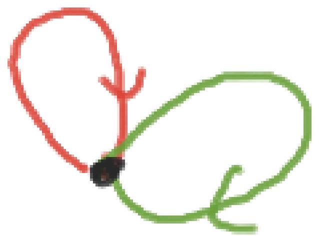
We know that a covering space of Γ is a graph.



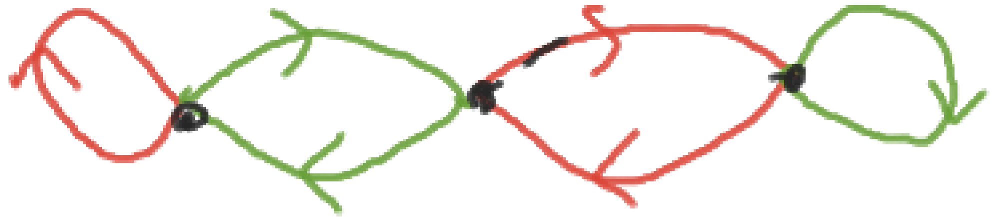
a bunch of copies of the same local situation over each vertex, edge.

over

<u>Examples:</u>



Covering:



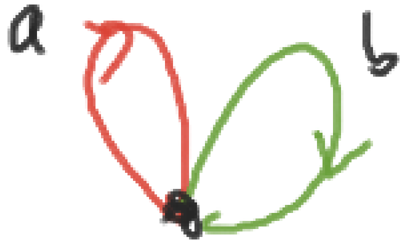
Universal covering of $X =$ 

($\{1, \dots, n\}$ can be replaced by an infinite set)

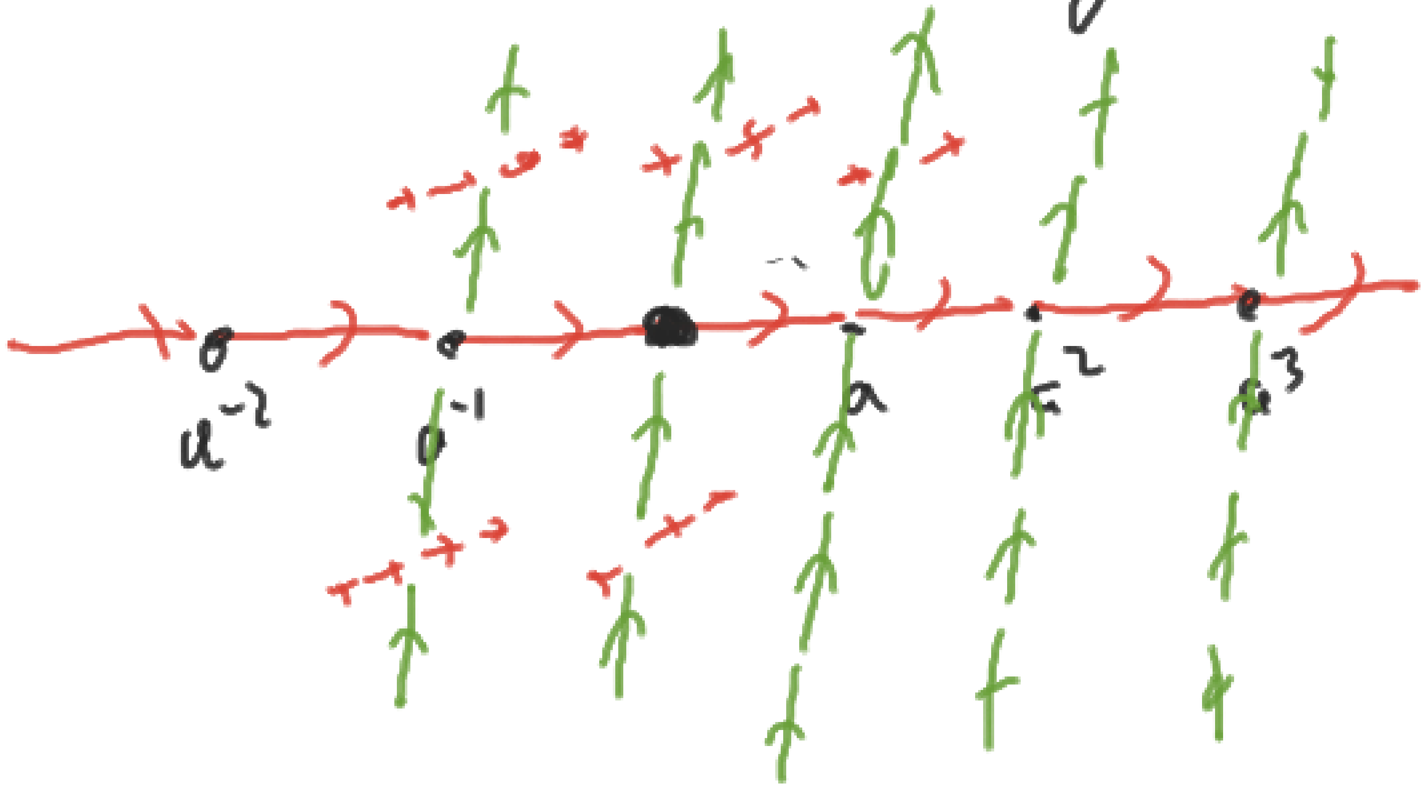
$\tilde{X}$: $\tilde{X}_0$ = simple words in $F(e_1, \dots e_n)$
(elements of $F(e_1, \dots e_n)$)

$$\tilde{X}_1 = \{ (w, we_i) \mid w \in \tilde{X}_0 \}$$

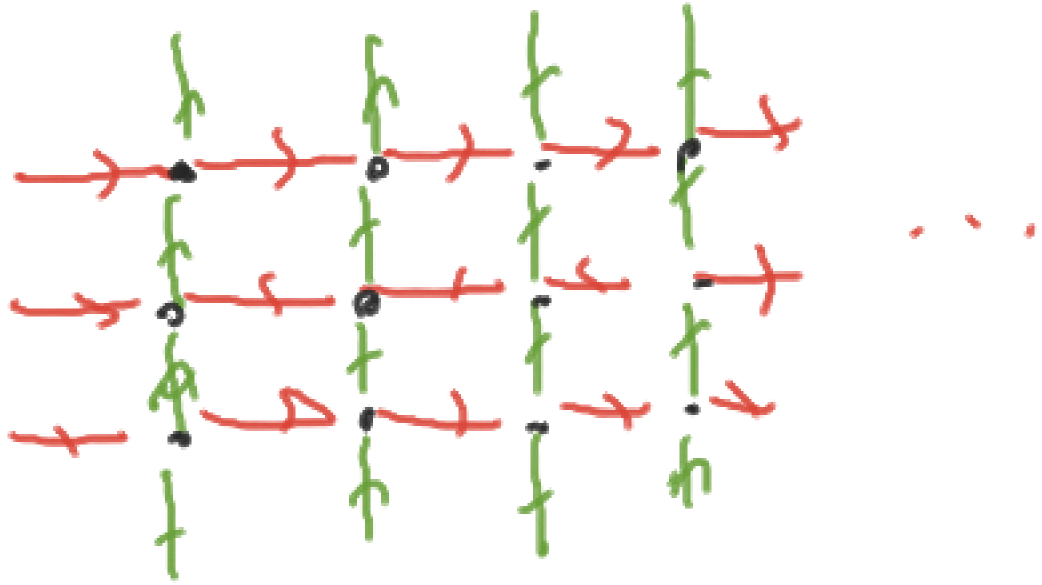
$\tilde{X}$ is a tree ∴ it is the universal covering of X.



Universal covering:



HW 5 Describe

$\pi_1\Big($  $\Big)$

Vertices: $\{(k,l) \mid k,l \in \mathbb{Z}\}$
$= \mathbb{Z}^2$

Edges $((k,l),(k+1,l))$, $k,l \in \mathbb{Z}$.
$((k,l),(k,l+1))$

Once we have the universal covering $\tilde{X}$ of a space X, we can construct a based covering corresponding to any subgroup $H \subseteq \pi_1(X,x)$: H acts on $\tilde{X}$ (freely), and take $\tilde{X}/H$.

This proves a

Theorem: A subgroup of a free group is free.

Proof: $F(S) = \pi_1(\bigvee_S S^1, *)$. For every $H \subset F(S)$ there is a covering (con.)

$$(\tilde{X}, \tilde{x}) \to \bigvee_S S^1$$

$$H = \pi_1(\tilde{X}, \tilde{x}) \subseteq \pi_1(\bigvee_S S^1, *).$$

$\tilde{X}$ is a graph. $\therefore$ H is a free group. $\square$