MATH 592 2/16/2024

Universal covering of an arbitrary connected graph Γ. Vertices V, edges E

$S, T : E \to V$

Select a spanning tree Θ. Let E_0 be the edges of Θ, E_1 be the edges not in Θ. $\quad (\pi_1(\Gamma, x_0) \cong F(E_1) = \langle E_1 \mid \rangle)$

x_0 ← vertex

Universal cover $\tilde{\Gamma}$: Vertices $V \times F(E_1)$. $\quad$ Edges: $\overset{e}{E_0} \times F(E_1)$

$p : \tilde{\Gamma} \to \Gamma$ (projection, forget w)

$$S(e, w) = (S(e), w)$$
$$T(e, w) = (T(e), w)$$

$\overset{f}{E_1} \times F(E_1) \quad S(f, w) = (S(f), w) \quad T(f, w) = (T(f), wf)$

For any subgroup $H \subseteq \pi_1(\Gamma, x_0)$, we get a covering $\bar{\Gamma}$ with a vertex $\bar{x}_0$ and $\pi_1(\bar{\Gamma}, \bar{x}_0) = H \subseteq \pi_1(\Gamma, x_0)$ by taking $\bar{\Gamma} = \tilde{\Gamma} / H$

acts on $\tilde{\Gamma}$ by acting on $p^{-1}(x_0)$

∴ **Theorem**: A subgroup of a free group is free.

Proof: A covering of a graph is a graph. □

Example: Suppose I have a free group $F(x_1,\dots,x_n)$. Suppose I have a subgroup $H \subseteq F(x_1,\dots,x_n)$ of index k. We know that H is free. What is the rank of H?

rank of the free group.

Note: Free groups of different rank are not isomorphic

$F(x_1,\dots,x_n)^{ab} = \langle x_1,\dots,x_n \rangle_{ab}$

$= \mathbb{Z}^n$.

Linear algebra,

$F(x_1,\dots,x_n) = \pi_1($  $, *)$

$\| \qquad\qquad \Gamma = \bigvee_n S^1$

G

Index k although H corresponds to a connected covering

$q : \tilde{\Gamma} \to \Gamma \qquad q^{-1}(*) = G/H \leftarrow$ has k elements

$q^{-1}(x_i)$ also consists of k edges.

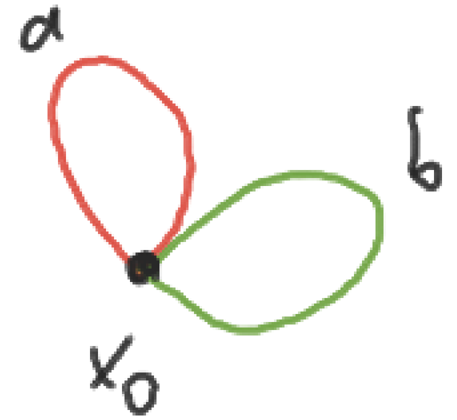
$\tilde{\Gamma}$ has k vertices, nk edges. It is connected (by construction).

Spanning tree: $(k-1)$ edges.

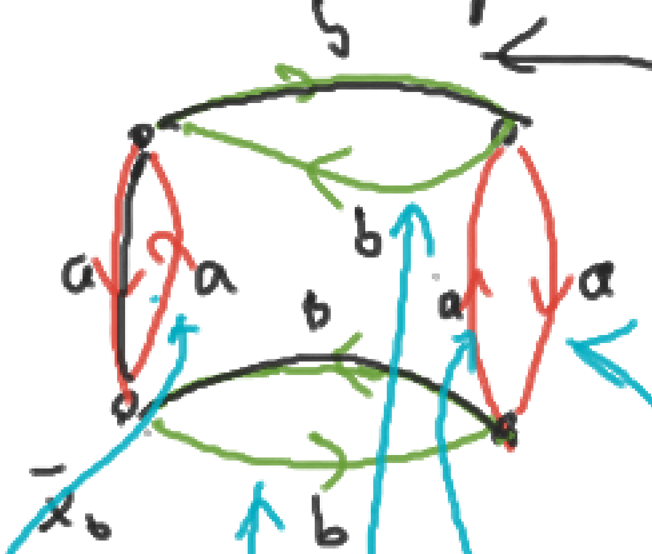
$\therefore$ $H = \pi_1(\tilde{\Gamma})$ has rank $\underline{\underline{nk - k + 1}}$.

<u>Example</u>: Find free generators of the subgroup H of $F_2 = \langle a, b \mid \rangle$ generated by squares of all words in a, b. Is H a normal subgroup of F_2?

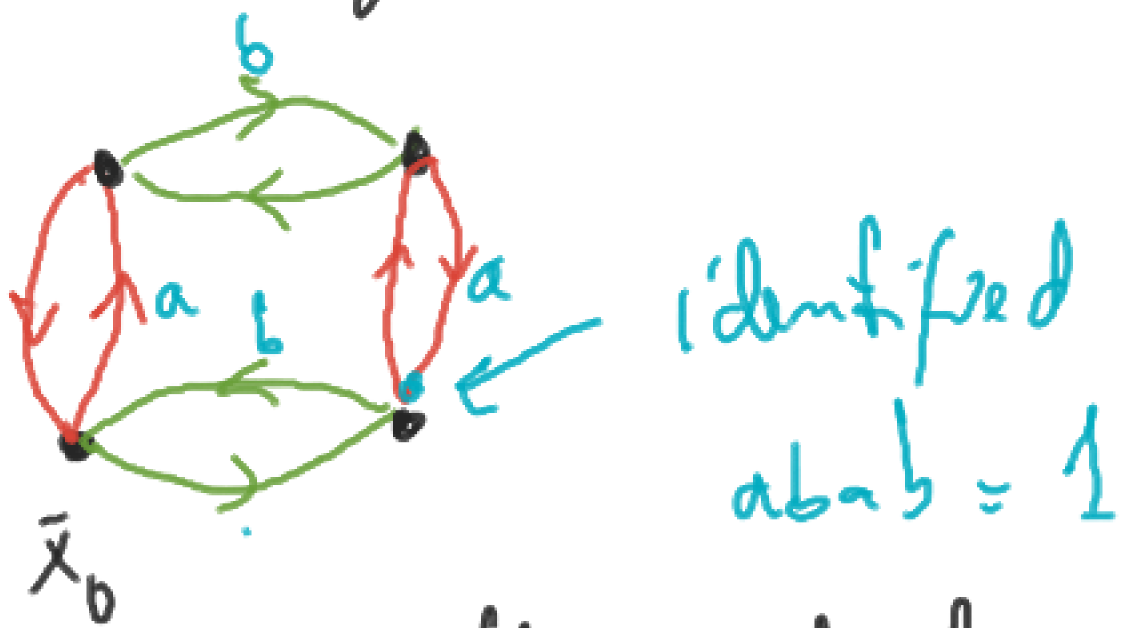
Constructing the connected covering corresponding to H: "casting out"



Select a spanning tree



one example of a spanning tree

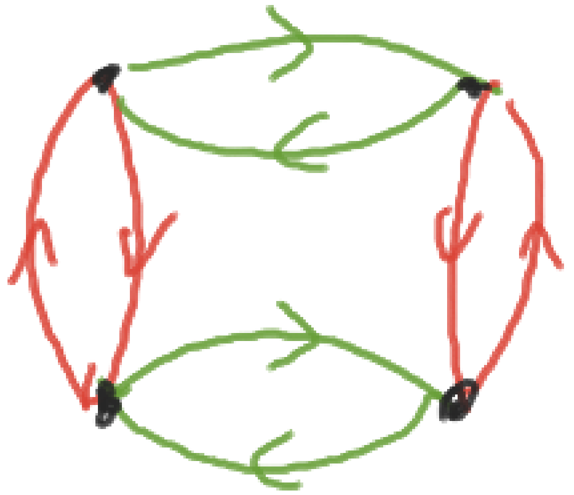


There is no further identification because this is a covering.

Edges not in the spanning tree determine the rank.

rank = 5. Words to write down: (trip in spanning tree to beginning of the edge). (the edge). (trip in spanning tree back to the origin)

$\boxed{a^2,\ b^2,\ a^{-1}b^2a,\ b^{-1}aba^{-1},\ a^{-1}bab}$ ← <u>Answer</u> (many other answers are possible).



Is the subgroup normal?

If a subgroup $H \triangleleft G$ is normal, then the automorphisms of G/H is G-Set is the G/H $\quad (N(H) = G)$.

This is equivalent to the group of automorphisms of G/H as a G-set acting transitively on G/H $\left(\forall \underbrace{x, y}_{\in G/H} \; \exists \varphi \quad \varphi x = y \right)$

$$\varphi : G/H \xrightarrow{\cong} G/H \in \mathrm{Mor}(G\text{-Set}).$$

In other words, a connected (based) covering $p:(\bar{X},\bar{x}_0)\to(X,x_0)$ of a connected CW-complex X corresponding to a subgroup $H\subseteq G$ is regular, which means that: $\boxed{\text{upstairs deck transformations which are isomorphisms}}$ act transitively of $p^{-1}(x_0)$

$N(H)=G$
$W(H)=G/H$

if and only if $H\triangleleft G$ is a normal subgroup.

This is a regular covering.
Yes, this subgroup is normal.

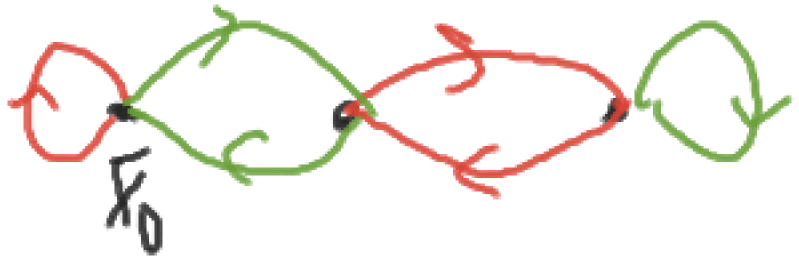
Automorphism = automorphism of graph, preserves arrows, colors.

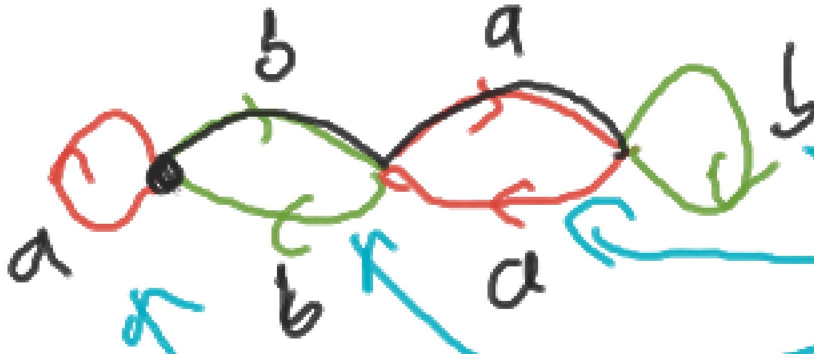
Can I get from any vertex to any other vertex?

An example of a covering of 

which is not regular:

$\bar{X}$

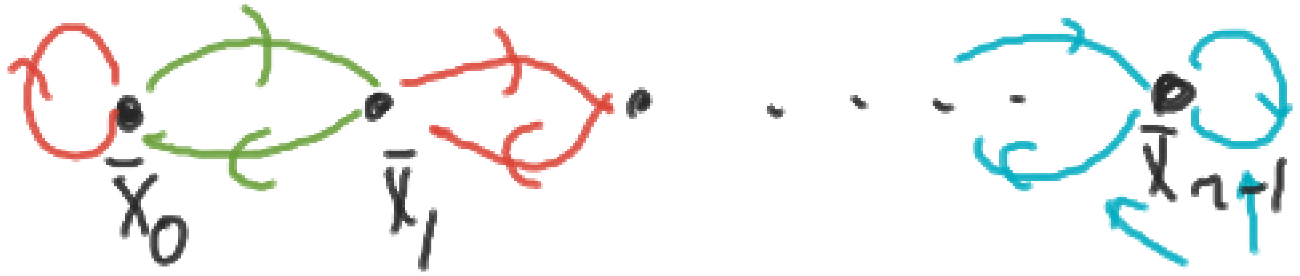




Free generators of $\pi_1(\bar{X}, \bar{x}_0)$:

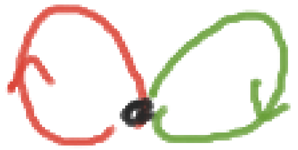
$\underbrace{a,\ b^2,\ ba^2b^{-1},\ baba^{-1}b^{-1}}$ rank $= 4$.

HW. 6



$\bar{X}(n)$

red or green depending on n even or odd.

is a covering of X . (a) Find out for which values of n is this covering $p_n : \bar{X}(n) \longrightarrow \bar{X}$ regular. (b) Find the rank and free generators of $\pi_1(\bar{X}(n), \bar{x}_0)$.

Preview: When $H \lhd \underbrace{F(a_1, \dots, a_n)}_{G}$, then the graph which is the corresponding connected covering to H is the Cayley graph of the group G/H.

vertices: element of G/H, edges: how the generators act by left translation.

Note: G/H, at least theoretically, is any group.

Tuesday discussion: Examples of this type.