MATH 592 2/19/2024

<u>Cayley graph</u> : G group, $a_1, \dots a_n \in G$ generate the group

$$C(G)\begin{cases} V = G \\ E = (g, ga_i) \quad g \in G \quad i = 1, \dots, n \end{cases}$$

We have a covering

$$p : C(G) \longrightarrow \bigvee_n S^1$$
$$a_i \longmapsto a_i$$

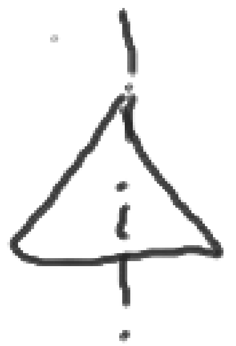
a_i ← based homeomorphism with the i-th cell of $\bigvee_n S^1$

Conversely, every regular covering $p : \Gamma \to \bigvee_n S^1$ is of this form where

Choose a base point $\tilde{x}_0$ in vertices of Γ

$$G = \underbrace{F(a_1, \dots a_n)}_{\pi_1\left(\bigvee_n S^1_1, *\right)} \Big/ \pi_1(\Gamma, \tilde{x}_0).$$

<u>Example</u>: Σ_3 - symmetric group (= all permutations) on three elements = isometries of an equilateral triangle. Generators:



a

reflection

b

rotation $\frac{2\pi}{3}$ clockwise

Find free generators of the kernel

$K = \ker(F(a,b) \to S_3)$

$a \longmapsto$ refl.

$b \longmapsto$ rotation

$n = 2$

$k = 6$

Free generators $a^2,\ b^3,\ b^{-2}a^2b^2,\ b^{-1}a^2b,\ aba^{-1}b,\ b^{-2}aba^{-1},\ b^{-1}aba^{-1}b^2$

Rank 7

Index k subgroup of F_n has rank $kn - k + 1$ $\left.\right\}$ $n = \#$ generators, $k = \# G$

k vertices kn edges

HW **1** Find free generators of the group $K = \ker\left(F(a,b) \to (\mathbb{Z}/4,+)\right)$ and also the rank of K.

$a \longmapsto 1$

$b \longmapsto 2$

Example: Find free generators of the subgroup of $F(\dot{a},\dot{b})$ generated by a, b^3, (all conjugates of a^2 — $w^{-1}a^2w \ \forall w$), bab^{-1}, $b^{-1}a^{-1}bab$,

$a, \ bab^{-1}, \ b^3, \ b^{-1}a^2b, \ b^{-1}aba^{-1}b$

rank 5

$n=2$

$k=4$

$8-4+1=5$ ✓

Is this subgroup normal? <u>NO</u> Not a regular covering.

HW (2) Let H be the subgroup of $F(a,b)$ generated by all conjugates of $a^2, b^2, a, babab$. Find free generators of H. Is H a normal subgroup? What is its rank?

<u>Coverings of a general connected CW-complex</u> X: Recall: X_1 1-skeleton is a graph

2-cells correspond to conjugacy classes in $\pi_1(X_1, x_0)$ where $x_0 \in X_0$. How do we construct a based covering $p: \tilde{X}_1 \to X_1$ corresponding to a subgroup $K \subseteq \pi_1(X_1, x_0)$? (Most interesting case: $K = \{e\}$ universal covering

<u>Recall from algebra</u>: Classification of subgroups of $K \subseteq G/H$ where $H \lhd G$.

double arrow means onto

$$\begin{tikzcd} K' \arrow[r, two heads] \arrow[d, phantom, sloped, "\subseteq"] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, phantom, sloped, "\subseteq"] \\ G \arrow[r, two heads] & G/H \end{tikzcd}$$

pull back

What subgroups $K' \subseteq G$ do we get as a result of such pullbacks?

<u>Answer</u>: We get precisely subgroups $K' \subseteq G$ such that $H \subseteq K'$.

(then automatically $H \lhd K'$)

The correspondence

$$\{\text{All subgroups } K' \subseteq G \text{ with } H \subseteq K'\} \longrightarrow \{\text{subgroups of } G/H\}$$

$$K' \longmapsto K'/H$$ is bijective.

More details and further examples in tomorrow's discussion.

How do we construct the covering of X_2 corresponding to (based connected)

$K = \pi_1(\tilde{X}_2, \tilde{x}_0) \subseteq G/H = \pi_1(X_2, x_0)$

$G = \pi_1(X_1, x_0)$, $H \trianglelefteq G$ generated by the conjugacy class corresponding to 2-cells?

Answer: Consider the pullback $K' \subseteq G$. Recall $K' \supseteq H$.

Construct the based cover $p_1 : \tilde{X}_1 \longrightarrow X_1$ corresponding to $K' \subseteq G$.

$$\tilde{x}_0 \longmapsto x_0$$

Since $K' \supseteq H$, all the attaching maps of 2-cells lift. Get $p_2 : \tilde{X}_2 \longrightarrow X_2$

by attaching a 2-cell to $\tilde{X}_1$ along each of these lifts.

If X cells of dimension > 2, then all their attaching maps will automatically lift, by attaching n-cells to all lifts of attaching maps, $n = 3, 4, \ldots$

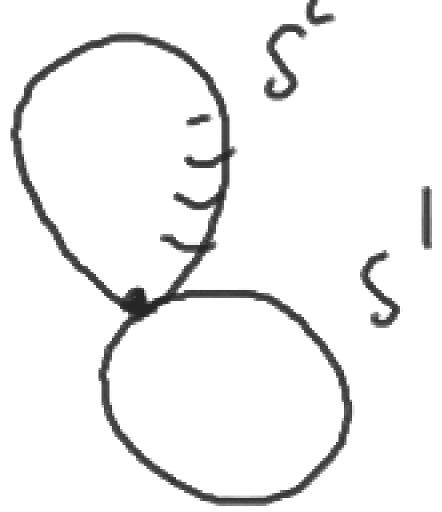
S^{n-1} is simply connected

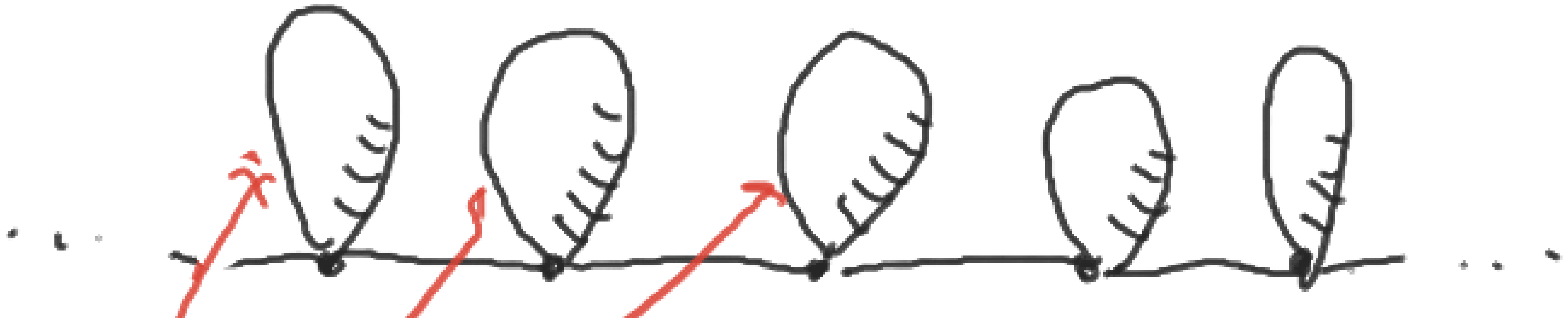
$\pi_1(S^{n-1}) = 0$,

$n-1 \geq 2$

<u>Example</u>: Find the universal covering of $S^2 \vee S^1$.

Solution:





all lifts of the 2-cell.

HW 3 Describe the universal cover of $\mathbb{RP}^2 \vee \mathbb{RP}^2$.