MATH 592 2/21/2024

<u>Example</u>: Universal covers of compact surfaces.

S^2 is simply connected. $(\pi_1(S^2)=0)$.

$T = S^1 \times S^1$. Universal cover: $\mathbb{R}\times\mathbb{R} \longrightarrow S^1\times S^1$

$$(s,t) \longmapsto (e^{2\pi i s}, e^{2\pi i t}).$$

What about $\underbrace{T\#\cdots\#T}_{n>1}$?

Hyperbolic plane geometry

can assume $= 1$

$$\frac{az+b}{cz+d}$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$$

$a,b,c,d \in \mathbb{C}$

composition corresponds to product of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{pmatrix} a\lambda & b\lambda \\ c\lambda & d\lambda \end{pmatrix}$$

give the same map

Möbius transformations. Holomorphic diffeomorphisms

$$\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$$

preserve orientation and <u>angles</u>

$$\frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{R} \qquad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

preserves the upper half-plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$.

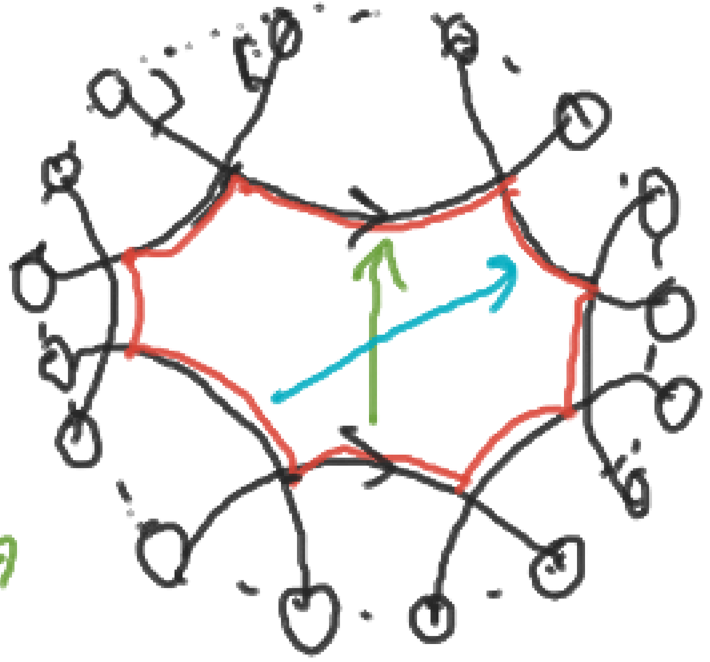
hyperbolic plane

the group of such transformations $PSL_2(\mathbb{R})$ is the group of hyperbolic isometries

We can put a Riemann metric which is indeed invariant, straight lines = ray or semicircle perpendicular to the real line.

$\longrightarrow$ shift in a real direction is a hyperbolic isometry. (*)

Applying a Möbius transformation, you can transform $\mathbb{H}$ into an open disk. Then the isometries of the open disk are examples of hyperbolic isometries.

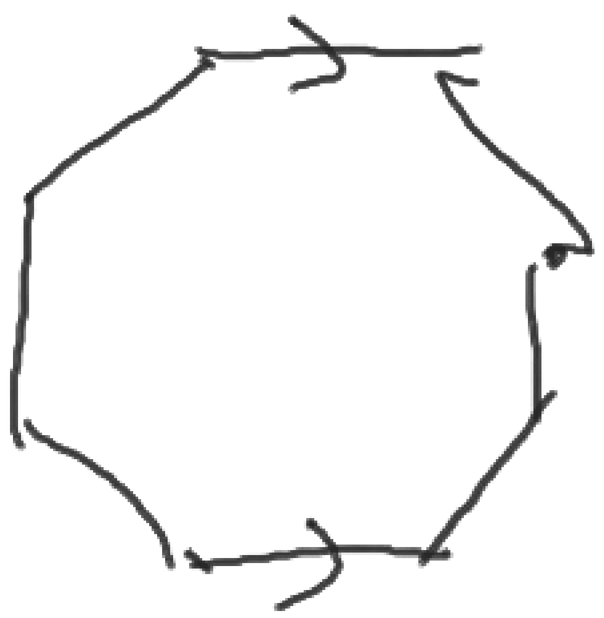


We can draw a hyperbolic regular $2n$-gon

invariant under rotation by $e^{2\pi i/2n}$

assuming this is ~~the~~ open unit disk $\mathbb{D}$.

A hyperbolic isometry "translation" (conjugate to $(*)$) sends a side to the opposite side in matching orientation. The subgroup Γ generated by all these translations acts freely on $\mathbb{D}$, $\mathbb{D}/\Gamma$ =



regular $2n$-gon with opposite sides identified matching orientation

(In HW, we proved that this is $\underset{[\frac{n}{2}]}{\#} T$.)

$\therefore$ The universal covering space of $\underset{n}{\#} T$, $n \geq 1$, is contractible

$\mathbb{R}^2$ $n=1$, $\mathbb{H}$ $n>1$.

Proposition: Let $p : \tilde{X} \longrightarrow X$, $\tilde{x}_0 \longmapsto x_0$ be a universal cover of a based CW-complex X. Then for all $n > 1$, the map

$$\pi_n p : \pi_n \tilde{X} \xrightarrow{\cong} \pi_n X$$

is an isomorphism.

Proof: S^n is simply connected for $n > 1$. So every $S^n \longrightarrow X$, $* \longmapsto x_0$ lifts and a null-homotopy also lifts. $\square$

Theorem: For $n \geq 1$, $k \geq 2$,

$$\pi_k(\underbrace{T \# \cdots \# T}_{n}) = 0. \quad \square$$

$= \pi_k(\text{universal cover})$

contractible

Are $\pi_k(S^2)$ known? $k \geq 2$

(The situation is unclear. Easy answer: NO. Some claims about 20 years ago giving some elegant algebraic description.)

Status?

Wu, F. Cohen

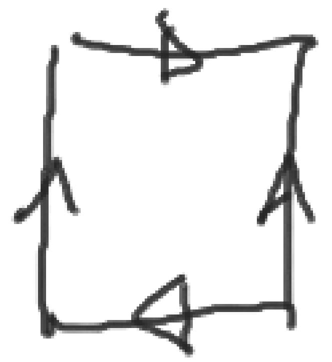
Non-orientable:

HW 4 Find the universal cover of $\mathbb{R}P^2$.

For $n > 2$, again, $\mathbb{H}$ can be made into the universal cover of $\#_n \mathbb{RP}^2$

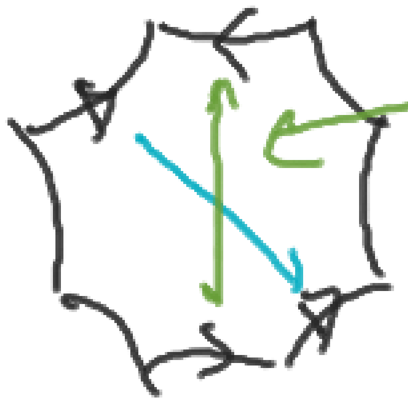
For $n = 2$: $\mathbb{R}^2$ is the univ. cover.

$\mathbb{RP}^2 \# \mathbb{RP}^2$



Klein bottle

2-fold cover is the torus.



unmatching orientation = translation composed with reflection of the disk.

Attach all opposite pairs with matching orientation except one, get $\#_n \mathbb{RP}^2$.

$\therefore$ Except $\mathbb{RP}^2$, S^2, all compact surfaces have contractible universal covers.

The orbit category of G = full subcategory of G-Set on objects which are orbits : G/H ($H \subseteq G$ subgroup).

Morphisms $G/H \longrightarrow G/K$ are determined by $g \in G$

$1H \longmapsto gK$ (only depend on $g \in G/K$)

Which $g \in G/K$ are allowed?

$$h \in H \longmapsto hgK \in gK, \quad \Leftrightarrow \quad g^{-1}hg \in K.$$

Morphisms $G/H \longrightarrow G/K$ correspond to $g \in G/K$ such that

$\boxed{g^{-1}Hg \subseteq K}$ ← subconjugacy.

Example: $G = \mathbb{Z}/2$.

G/H		
$H = \mathbb{Z}/2$	$G/H = *$	$\mathcal{O}_{\mathbb{Z}/2}$
$H = \{e\} = \{0\}$	$G/\{0\} = \cdot \circlearrowright \cdot$ generator of $\mathbb{Z}/2$	$\mathcal{O}_{\{0\}}$

We can draw the orbit category

$$\begin{array}{c} \mathcal{O}_{\mathbb{Z}/2} \circlearrowleft \mathrm{Id} \\ \uparrow \\ \mathrm{Id} \circlearrowright \mathcal{O}_{\{0\}} \circlearrowleft \text{generator} \in \mathbb{Z}/2 \end{array}$$

Different elements of $G/\{0\}$.

$\mathcal{O}_{\mathbb{Z}/2} \to \mathcal{O}_{\mathbb{Z}/2}$

$\mathcal{O}_{\{0\}} \to \mathcal{O}_{\mathbb{Z}/2}$

Any $g \in \mathbb{Z}/2$ would work, but we are in $\underbrace{\mathbb{Z}/2/\mathbb{Z}/2}= *$

HW **5** Describe completely the orbit category of $\mathbb{Z}/4$.

Example:

Note: For finite groups, every morphism of orbits $G/H \to G/H$ is an automorphism.

$G = \Sigma_3 = \underbrace{\text{symmetric group}}_{\text{all permutations}}$ on three elements

Automorphism group $= W(H) = NH/H$

Describe the orbit category up to equivalence. (the skeleton)

$$\begin{tikzcd}
& \Sigma_3/\Sigma_3 \arrow[loop above, "\mathrm{Id}"] & \\
\Sigma_3/\mathbb{Z}/2 \arrow[ur] \arrow[loop left, "\mathrm{Id}"] & & \Sigma_3/\mathbb{Z}/3 \arrow[ul] \arrow[loop above, "\mathbb{Z}/2"] \\
& \Sigma_3/\{e\} \arrow[ul] \arrow[uu] \arrow[ur, shift left] \arrow[ur, shift right] \arrow[loop below, "\Sigma_3"] &
\end{tikzcd}$$

reflection

rotation

← automorphism group

Orbits by conjugate subgroups are isomorphic.