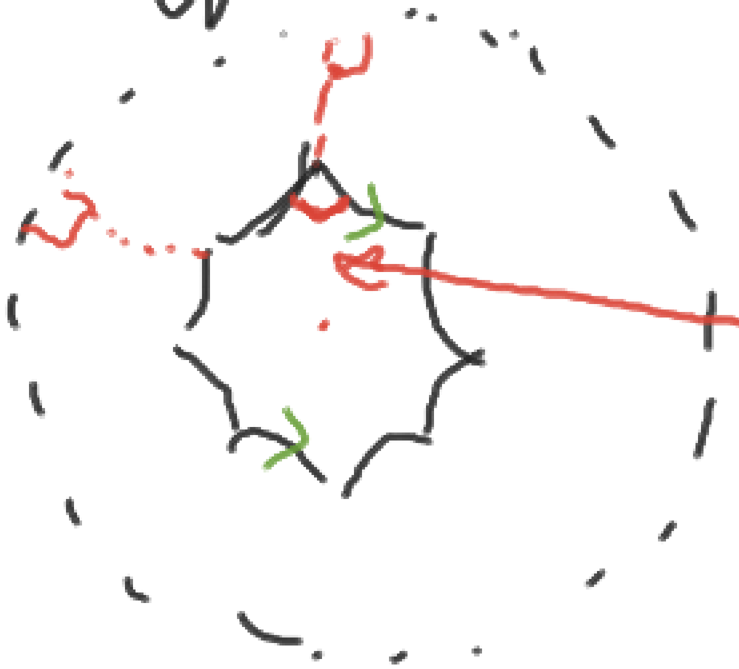
MATH 592 2/23/2024

Correction of the hyperbolic example:



regular $2n$-gon

$<$ what it would be in the Euclidean case (any $<$ works).

- The angles need to be $\frac{2\pi}{2n}\left(=\frac{\pi}{n}\right)$ (2n equal angles to each vertex)
- n has to be even. (if n is odd, the action is not free)

For the unoriented case, given a smooth structure on the compact surface. An unoriented connected smooth manifold has a connected double cover (remembering the orientation).

2-sheeted

$\# p^{-1}(x) = 2$

Some examples of the equivalence of categories between the orbit category of $\pi_1(X, x)$ and the category of connected covering spaces and (unbased) deck transformation.

Example: By choosing a presentation of Σ_3, we can construct a connected CW-complex X, $x_0 \in X_0$, $\pi_1(X, x_0) \cong \Sigma_3$. What isomorphism classes of covering spaces does X have, and which ones are regular coverings? More generally, what automorphisms do they have?

Solution:

universal cover

$\tilde{X}_6 \circlearrowleft \Sigma_3$

4 iso classes of connected coverings

regular

not regular

$\tilde{X}_3$

$\tilde{X}_2 \circlearrowleft \mathbb{Z}/2$

regular

$\{e\} \circlearrowleft X$

regular

the subscript denotes the number of sheets (= degree) of the covering

Orbit category

$$\begin{tikzcd}
 & \Sigma_3/\{e\} \circlearrowleft \Sigma_3 \arrow[dl] \arrow[dr] & \\
\Sigma_3/\mathbb{Z}/2 \arrow[dr] & & \Sigma_3/\mathbb{Z}/3 \circlearrowleft \mathbb{Z}/2 \arrow[dl] \\
 & \Sigma_3/\Sigma_3 &
\end{tikzcd}$$

= index of the subgroup

the subgroup corresponds to π_1 of the covering

A more geometric

Example: Describe all isomorphism classes of degree 2 coverings of $T = S^1 \times S^1$.

Solution: $\pi_1(T, x_0) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$

? index 2 subgroups

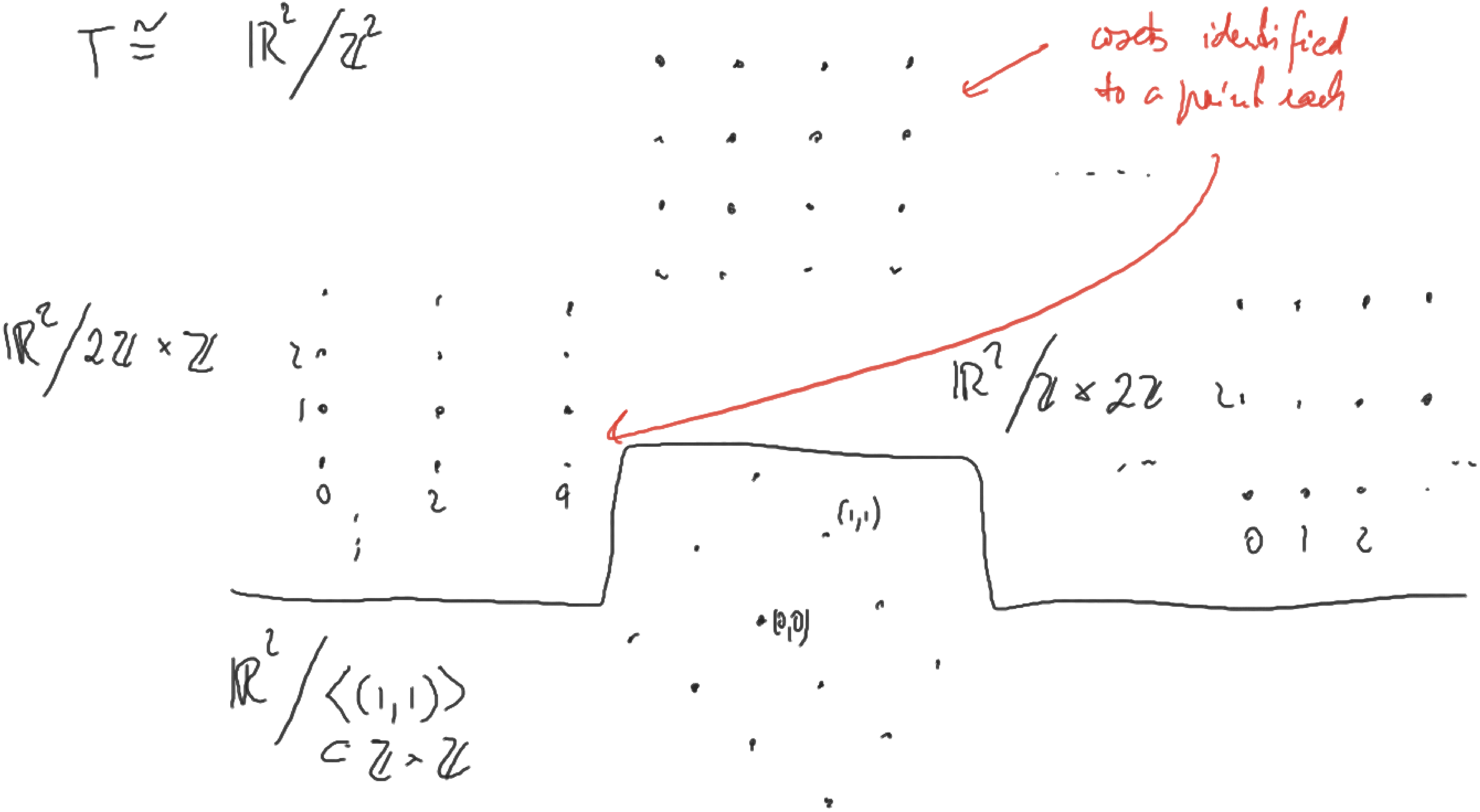
$$\begin{array}{ccccc} \mathbb{Z} & \oplus & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \\ a & & b & & \{0,1\} \end{array}$$

3 possibilities:

$a \longmapsto 0$ $\quad$ $a \longmapsto 1$ $\quad$ $a \longmapsto 1$

$b \longmapsto 1$ $\quad$ $b \longmapsto 0$ $\quad$ $b \longmapsto 1$

Answer: 3 isomorphism classes of degree 2 coverings



$T \cong \mathbb{R}^2/\mathbb{Z}^2$
cosets identified to a point each
$\mathbb{R}^2/2\mathbb{Z}\times\mathbb{Z}$
2
1
0
2
4
$\mathbb{R}^2/\mathbb{Z}\times 2\mathbb{Z}$
2
0
1
2
(1,1)
(0,0)
$\mathbb{R}^2/\langle(1,1)\rangle \subset \mathbb{Z}\times\mathbb{Z}$

HW 6 Describe all the (unbased) isomorphism classes of connected covering of $\mathbb{R}P^2 \times \mathbb{R}P^2$

What is next? The fundamental group is not enough.

Example: Prove that there does not exist a continuous map $D^n \to S^{n-1}$ which would be the identity on $S^{n-1} \subseteq D^n$.

$$\begin{tikzcd} D^n \arrow[r] & S^{n-1} \\ S^{n-1} \arrow[u, phantom, "\cup\!|" rotate=90] \arrow[ur, "\mathrm{Id}"'] & \end{tikzcd}$$

$\Leftarrow$ S^{n-1} is not a retract of D^n.

Solution for $n=2$: Functors preserve retracts

$$\begin{tikzcd} 0 = \pi_1(D^2) \arrow[r] & \pi_1(S^1) = \mathbb{Z} \\ \mathbb{Z} = \pi_1(S^1) \arrow[u] \arrow[ur, "\mathrm{Id}"'] & \end{tikzcd}$$

contradiction

This proof with π_1 won't work for higher n.

<u>Theorem</u> (Hopf): $\pi_k(S^k) \cong \mathbb{Z}$. (Proof: later)

So we could use π_{n-1}.

π_n are much harder to compute than π_1.

We need a computable tool – <u>homology</u>.

Example: For a connected CW-complex X, $H_1(X) = \pi_1(X,x)^{Ab}$

base point does not matter when we abelianise.

$\boxed{\mathbb{Z}S = \text{free abelian group on a set } S \ (= FS^{Ab})}$

$\langle S \mid \ \rangle_{ab}$

$e_1 \mapsto v_0 - v_1$

spanning tree Γ of X,

set of k-cells of X : I_k

$E = I_1 \setminus$ edges in Γ

$\pi_1(X)^{Ab} = \mathbb{Z}E \,/\,$ relations by 2-cells

(suppress edges in Γ)

More symmetrical view: $\mathbb{Z}I_1 \xrightarrow{S-T \,=\, \partial_1} \mathbb{Z}I_0$

$\underset{\pi_1(X_1)^{Ab}}{\overset{\shortparallel}{\mathbb{Z}E}} \cong \ker S-T : \mathbb{Z}I_1 \to \mathbb{Z}I_0$

$H_1X =_{df} \pi_1(X)^{Ab} = \mathbb{Z}E / \mathrm{Im}\,\partial$

$\mathbb{Z}I_2 \xrightarrow[\partial_2]{} \mathbb{Z}E$ ← relations corresponding to each 2-cell

$$\mathbb{Z}I_2 \xrightarrow[\partial_2]{} \mathbb{Z}I_1 \xrightarrow[\partial_1]{} \mathbb{Z}I_0 \qquad \partial_1 \circ \partial_2 = 0$$

write abelian groups additively when discussing homology

$$H_1(X) = \operatorname{Ker}\partial_1 / \operatorname{Im}\partial_2$$

What if we could make a generalization!

A <u>chain complex C</u> is a system of abelian groups C_n, $n \in \mathbb{Z}$ (0 when not defined) together with homomorphisms $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$.

Then we can define the <u>n'th homology</u>

$$H_n C := \operatorname{Ker}(d_n : C_n \to C_{n-1}) / \operatorname{Im}(d_{n+1} : C_{n+1} \to C_n)$$

For a CW-complex X with set I_k of all k-cells, we would like an <u>explicit</u> chain complex

$$\cdots \longrightarrow \mathbb{Z}I_n \xrightarrow{\partial_n} \mathbb{Z}I_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_0} \mathbb{Z}I_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

and then

$$H_n X = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$$ ← the homology of this chain complex.

Singular homology