MATH 592 3/4/2024

Write abelian groups additively. <u>Classification of finitely generated abelian groups.</u>

multiplicatively: $x^8 y^{20}, x^{15} y^6$

<u>Example</u>: $\langle x, y \mid 8x + 20y, 15x + 6y \rangle_{ab}$

$$\begin{pmatrix} 8 & 20 \\ 15 & 6 \end{pmatrix}$$

rows = relations

row-column diagonalisation over $\mathbb{Z}$. Elementary row operations:

- multiply row by ± 1
- add a $\mathbb{Z}$ multiple of one row to another.

Elementary column operations: do the same thing with columns.

This won't change the group up to $\cong$. Row operations give equivalent relations. Column ops. define the same relations in terms of other (equivalent) generators.

Doing row and column operations, we can get a matrix:

$$\begin{array}{c} \begin{array}{cccc} & & & \end{array} \overbrace{\begin{array}{ccc} & & \end{array}}^{\ell} \\ \left(\begin{array}{cccc|ccc} m_1 & & & & & & \\ & m_2 & & & & 0 & \\ & & \ddots & & & & \\ & & & m_k & & & \\ & & & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 0 \end{array}\right) \end{array}$$

$0 < m_1 \mid m_2 \mid \cdots \mid m_k$

$m_i \in \mathbb{N}$.

Relations in this form give the group

$\mathbb{Z}/m_1 \times \mathbb{Z}/m_2 \times \cdots \times \mathbb{Z}/m_k \times \mathbb{Z}^\ell$.

We can delete 0 rows but not $\underbrace{\text{0 columns}}_{\text{copies of } \mathbb{Z}}$!

$m_1 \mid m_2 \mid \dots \mid m_k$

forward

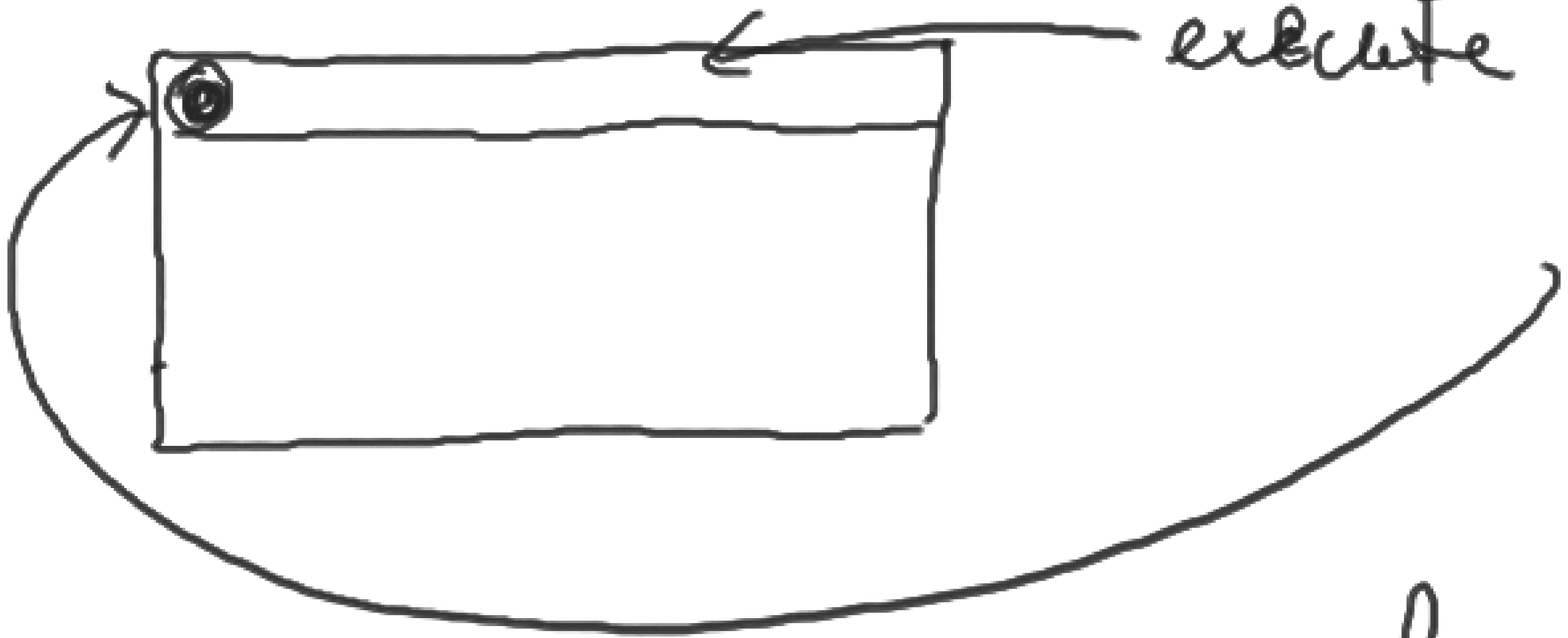
$$\boxed{\mathbb{Z}/m_1 \times \mathbb{Z}/m_2 \times \dots \times \mathbb{Z}/m_k} \times \mathbb{Z}^\ell \leftarrow \text{Smith form}$$

ℓ = rank of the ab. group.

we can delete any of these numbers which are $= 1$.

WOLOG, none are $= 1$. Then it uniquely characterises the group.

(counting elements of a given order).



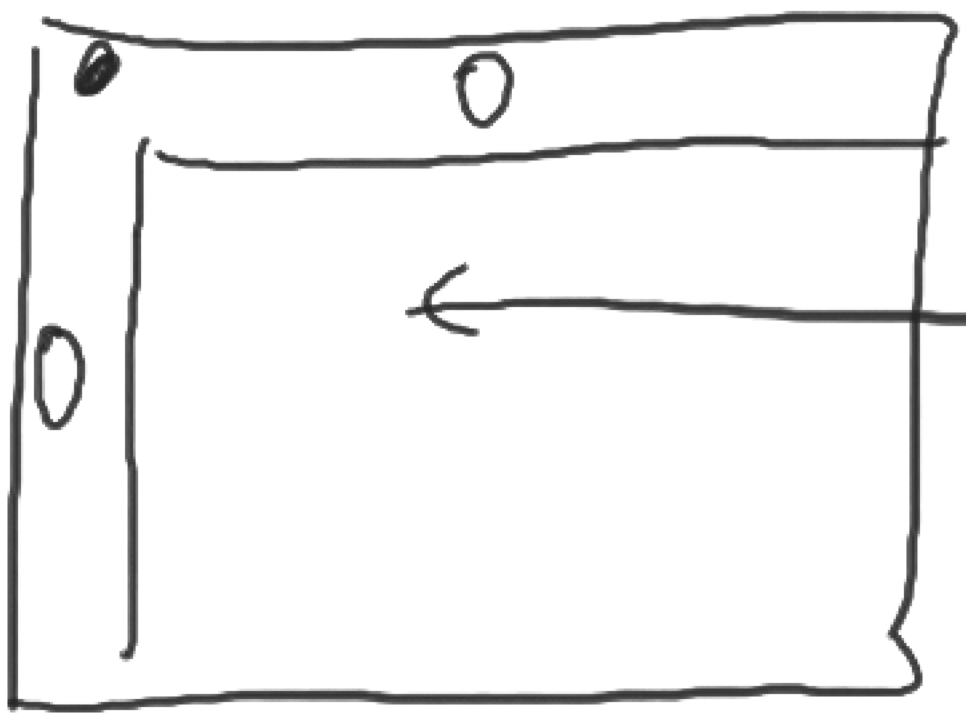
execute Euclidean algorithm to get pivot to divide everything in the first row.

clear the first row.

you need the pivot to divide every entry of the matrix.
(add it to a row. Repeat)

$\rightsquigarrow$

pivot (WLOG > 0) divides every other entry



repeat.

Example ⊛ :

$$\underset{-2}{\begin{pmatrix} 8 & 20 \\ 15 & 6 \end{pmatrix}} \sim \begin{pmatrix} 8 & 4 \\ 15 & -24 \end{pmatrix} \sim \underset{-2}{\begin{pmatrix} 4 & 8 \\ -24 & 15 \end{pmatrix}} \sim \begin{pmatrix} 4 & 0 \\ -24 & 63 \end{pmatrix} 6$$

$$\sim \begin{pmatrix} 4 & 0 \\ 0 & 63 \end{pmatrix} 1 \sim \underset{-16}{\begin{pmatrix} 4 & 0 \\ 4 & 63 \end{pmatrix}} \sim \begin{pmatrix} 4 & -64 \\ 4 & -1 \end{pmatrix} \sim \begin{pmatrix} -64 & 4 \\ -1 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 4 \\ -64 & 4 \end{pmatrix} 2 \cdot -1 \sim \overset{4}{\begin{pmatrix} 1 & -4 \\ -64 & 4 \end{pmatrix}} \sim 64 \begin{pmatrix} 1 & 0 \\ -64 & -252 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 252 \end{pmatrix}$$

Answer: $\mathbb{Z}/252$ (No $\mathbb{Z}$. $\mathbb{Z}$ comes from a $\underline{\underline{0 \text{ column}}}$)

Another form of f.g. ab. group classification:

$n = p_1^{k_1} \cdots p_m^{k_m}$

p_i different primes

$$\mathbb{Z}/n \cong \mathbb{Z}/p_1^{k_1} \times \cdots \times \mathbb{Z}/p_m^{k_m}$$

Chinese remainder theorem

$\oplus$ ← bi-product

$\lim(A \quad B)$ $\quad$ $\operatorname{colim}(A \quad B)$

HW ① Prove that $A \times B$ is both the product and coproduct of A, B in the category Ab of abelian groups.

A f.g. abelian group is also expressible uniquely (up to order of factors) as

$$\mathbb{Z}/p_1^{k_1} \oplus \cdots \oplus \mathbb{Z}/p_m^{k_m} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\ell}$$

$\mathbb{Z}^\ell$

$k_i \geq 1$, p_i primes, possibly repeating

primary cyclic groups

$$252 = 2\cdot 2\cdot 7\cdot 3\cdot 3 \qquad \mathbb{Z}/252 = \underbrace{\mathbb{Z}/4 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}/9}$$

$$126$$

$$63$$

Example: Classify $\langle a,b,c \mid 9a+3b+6c,\ 12a+3b+3c\rangle_{ab}$

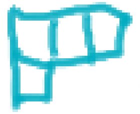
$$\begin{pmatrix} 9 & 3 & 6 \\ 12 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 3 & 9 & 6 \\ 3 & 12 & 3 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 0 \\ 3 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

Answer: $\mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}$.

HW ② Classify $\langle a, b, c, d \mid 2a+b+d,\ a+2b+c,\ b+2c+d,\ a+c+2d \rangle_{ab}$.

③ Classify $\langle a, b, c \mid 6a+15b+9c,\ 4a+12b+8c \rangle_{ab}$.

What rectangles can be filled by the L-pattern.

① (Above group related to . $\longleftarrow$ $\mathbb{Z}\{$squares in square ruled paper$\}$ / all patterns congruent to $\}$

$\underbrace{\text{The rectangle} \overset{?}{\in} \text{group}}_{\text{Necessary condition}}$. Is this condition sufficient for a large enough rectangle? Yes for . $(n=3)$.

In general, this question is unknown general n.

Chain complex C: C_k abelian groups $k \in \mathbb{Z}$ (not written means 0)

C: $\cdots \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} C_{k-2} \longrightarrow \cdots$

$\boxed{d_k \circ d_{k+1} = 0}$. Homology: $H_k C = \operatorname{Ker} d_k / \operatorname{Im} d_{k+1}$.

Often, $C_k = \mathbb{Z} I_k$ = free abelian group on set I_k. Suppose I_k finite. How do we use the above algorithm to calculate $H_k C$ for I_k finite?

matrices, work over $\mathbb{Q}$ (or $\mathbb{R}$)

__Answer__: • $\operatorname{rank} H_k C = |I_k| - \operatorname{rank} d_k - \operatorname{rank} d_{k+1}$

• __torsion__: Same as the torsion of $\mathbb{Z} I_k / \operatorname{Im} d_{k+1}$

"do ignore the $\mathbb{Z}$'s!" you got them from

Example: Find the homology of

$$C: \quad \mathbb{Z}^2 \xrightarrow[d_2]{\begin{pmatrix} 3 & -4 \\ 2 & 0 \\ -6 & 2 \\ 0 & -2 \end{pmatrix}} \mathbb{Z}^4 \xrightarrow[d_1]{\begin{pmatrix} 4 & 6 & +4 & -4 \end{pmatrix}} \mathbb{Z}$$

rank is 2

rank is 1

$$\begin{pmatrix} 4 & 6 & 4 & -4 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 2 & 0 \\ -6 & 2 \\ 0 & -2 \end{pmatrix} = \quad = \begin{pmatrix} 0 & 0 \end{pmatrix} \checkmark$$

torsion $=0$

$\operatorname{rank} H_2 = 0$

$2-2$

$H_1 C = \mathbb{Z} \oplus \mathbb{Z}/2$

$\operatorname{rank} H_1 = 4 - 2 - 1 = 1$

$\operatorname{rank} H_0 = 1 - 1 = 0$

torsion: $\mathbb{Z}/2$

$H_0(C) = \mathbb{Z}/2$

$$\begin{pmatrix} 4 \\ 6 \\ 4 \\ -4 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad H_2 C = 0$$

$$\begin{pmatrix} 3 & 2 & -6 & 0 \\ -4 & 0 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 & -6 & 0 \\ -1 & 2 & -4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 4 & 2 \\ 3 & 2 & -6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 4 & 2 \\ 0 & 8 & -18 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \quad \text{torsion } \mathbb{Z}/2$$

HW 4 Find the homology of the chain complex:

$$\mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 4 & 3 & 6 \\ 2 & 3 & 4 \\ 6 & 6 & 10 \\ 2 & 6 & 6 \end{pmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} -2 & -10 & 4 & 2 \\ -3 & -15 & 6 & 3 \end{pmatrix}} \mathbb{Z}^2$$