MATH 592 3/6/2024

Homological algebra (in abelian groups)

Exact sequence:

$$\cdots \to A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \qquad n \in \mathbb{Z} \text{ or } \mathbb{Z} \cap [a,b]$$

$$\operatorname{Ker} f_n = \operatorname{Im} f_{n+1}.$$

(Almost the same as a chain complex with homology 0, except when an exact sequence ends, no condition at first or last term. if applicable)

Examples:

$0 \to A \xrightarrow{f} B$ means: f injective

$A \xrightarrow{g} B \to 0$ means: g onto

$0 \to A \xrightarrow{h} B \to 0$ means: h is an isomorphism

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\subseteq", "i"'] & B \arrow[r, two heads, "j"'] & C \arrow[r] & 0 \end{tikzcd}$$

short exact sequence

$$\operatorname{Ker} j = \operatorname{Im} i$$

"$C = B/A$"

Chain : The category of chain complexes and chain maps:

$C \xrightarrow{f} D$: sequence of homomorphisms $f_n : C_n \longrightarrow D_n$

$$\begin{tikzcd} \cdots \arrow[r] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_n \arrow[r, "d_n"] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

$$\boxed{\forall n \in \mathbb{Z} \quad d_n \circ f_n = f_{n-1} \circ d_n}$$

$Z_n := \operatorname{Ker} d_n$

$B_n := \operatorname{Im} d_{n+1}$

Note: H_n : Chain $\to$ Ab is a functor.

$\overset{=f_*}{H_n f} : H_n C \to H_n D$

$f_n : Z_n^C \to Z_n^D$

$B_n^C \to B_n^D$

Short exact sequence of chain complexes:

$$0 \to C \xrightarrow{i} D \xrightarrow{j} E \to 0$$

i, j chain maps, for each n: $0 \to C_n \xrightarrow{i_n} D_n \xrightarrow{j_n} E_n \to 0$ is a short exact sequence. ("$E = D/C$")

⊛ **Theorem:** For a short exact sequence of chain complexes

$$0 \to C \xrightarrow{i} D \xrightarrow{j} E \to 0$$

we have a (long) exact sequence in homology:

$$\cdots \to H_n C \xrightarrow{H_n i} H_n D \xrightarrow{H_n j} H_n E \xrightarrow{\partial} H_{n-1} C \xrightarrow{H_{n-1} i} H_{n-1} D \xrightarrow{H_{n-1} j} \cdots$$

Addendum: ∂ is natural.

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, "f"'] & D \arrow[r] \arrow[d, "g"'] & E \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & 0
\end{tikzcd}$$

chain maps, rows exact.

We have a commutative diagram

$$\begin{tikzcd}
H_n E \arrow[r, "\partial"] \arrow[d, "H_n h"'] & H_{n-1} C \arrow[d, "H_{n-1} f"] \\
H_n E' \arrow[r, "\partial"] & H_{n-1} C
\end{tikzcd}$$

Examples of applications: If $0 \to C \to D \to E \to 0$ is a SES of chain cxes,

short exact sequence

If $H_n E = 0 \ \forall n \in \mathbb{Z}$ then $H_n C \cong H_n D$.

$H_n C = 0 \ \forall n \in \mathbb{Z} \Rightarrow H_n D \cong H_n E$

$H_n D = 0 \ \forall n \in \mathbb{Z} \Rightarrow H_n E \cong H_{n-1} C$

Snake lemma: If I have a diagram (commutative) of abelian groups

The columns are chain cxes.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

rows exact. Then we have a LES:

$$0 \to \mathrm{Ker}\, f \to \mathrm{Ker}\, g \to \mathrm{Ker}\, h \to \mathrm{Coker}\, f \to \mathrm{Coker}\, g \to \mathrm{Coker}\, h \to 0.$$

$$(\mathrm{Coker}\, f := A'/\mathrm{Im}\, f).$$

$$\begin{aligned} &\ \ \| \\ &\mathrm{Coeq}(f,0) = \mathrm{Colim}\Big(A \overset{f}{\underset{0}{\rightrightarrows}} A'\Big) \end{aligned}$$

HW ⑤ Prove that for homomorphisms of abelian groups

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

we have a long exact sequence:

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} g\circ f \to \operatorname{Ker} g \to \operatorname{Coker} f \to \operatorname{Coker} g\circ f \to \operatorname{Coker} g \to 0.$$

⑥ Split short exact sequence: Prove that the following are equivalent for a short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ of abelian groups:

ⓐ There exists a homomorphism $s: C \to B$ with $js = \mathrm{Id}_C$

ⓑ There exists a homomorphism $r: B \to A$ with $ri = \mathrm{Id}_A$

ⓒ We have a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\mathrm{Id}"'] & B \arrow[r, "j"] \arrow[d, "\cong"'] & C \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & A \arrow[r, "\mathrm{incl.}"] & A\oplus C \arrow[r, "\mathrm{proj}"] & C \arrow[r] & 0
\end{tikzcd}$$

$\mathrm{incl}(a) = (a,0)$

$\mathrm{proj}(a,c) = c$

Special case of snake lemma:

"9-lemma"

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

Columns exact.

2/3 rows exact, then the third one is exact

(Noether isomorphism theorem).

The 5-lemma:

$$\begin{tikzcd}
{} \arrow[r] & \bullet \arrow[r] \arrow[d, "a"'] & \bullet \arrow[r] \arrow[d, "\cong b"'] & \bullet \arrow[r] \arrow[d, "c"'] & \bullet \arrow[r] \arrow[d, "\cong d"'] & \bullet \arrow[r] \arrow[d, "e"'] & {} \\
{} \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & {}
\end{tikzcd}$$

rows exact. If b, d are isomorphisms, e injective, a onto ⇒ c is an isomorphism.

Proof of the LES:

? $x = dt$

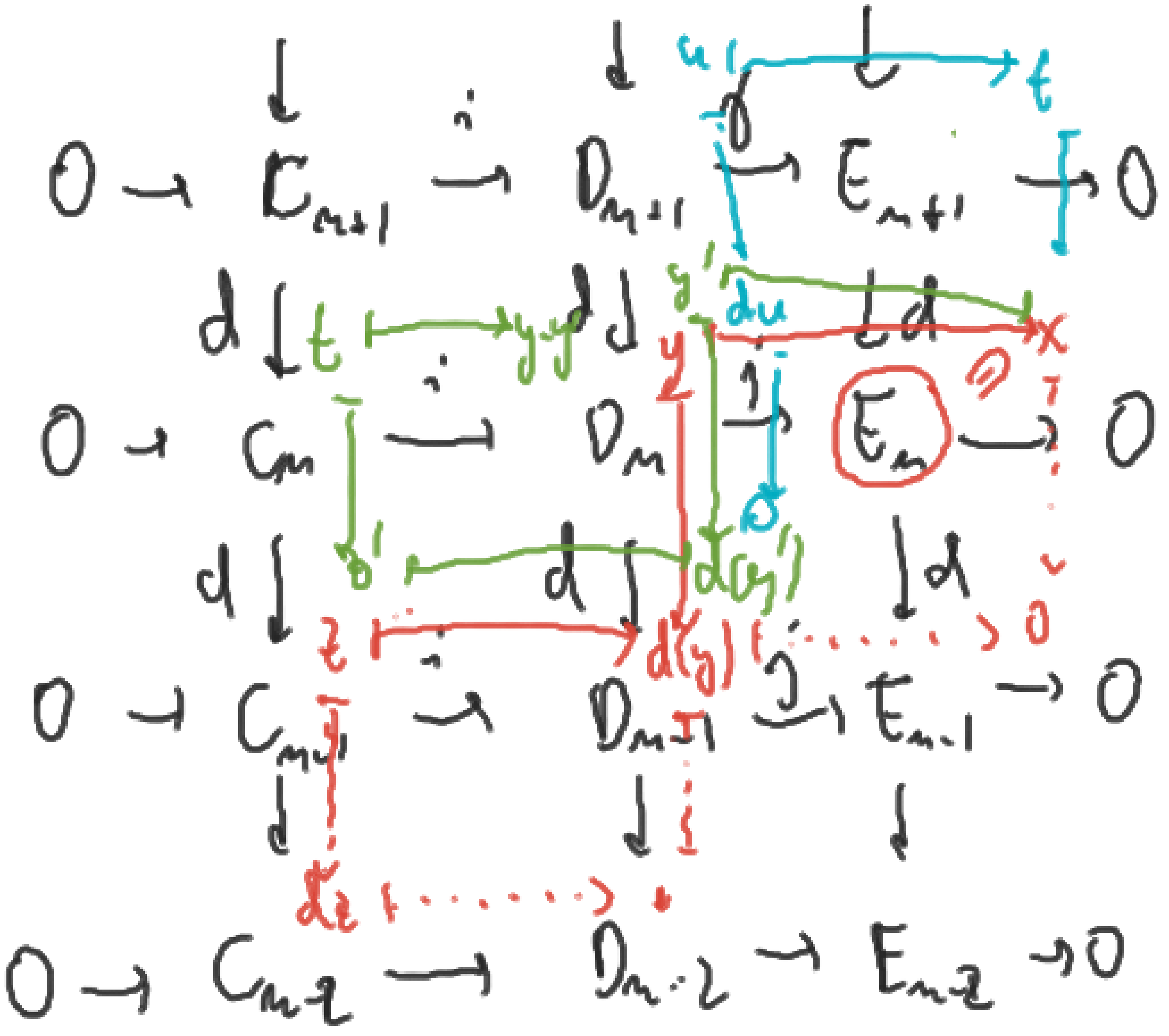
let $j(u) = t$

(j onto)

$jdu = dju = x$

$ddu = 0$

∴ get 0.



Defining ∂ : $[x] \in H_n E$

$dx = 0$

$[x] = [x + dy]$

$x = j(y)$ j onto

$jd(y) = dj(y) = dx = 0$

$\exists !\, z$ $i(z) = d(y)$

Definition attempt: $\partial [x] = [z]$

? $dz = 0$: $i\,d(z) = d\,i(z) = dd\,y = 0$

But i is injective. ✓

? $y' \mapsto x$ $y - y' \mapsto 0$ $\exists\, t$ $i(t) = y - y'$

i injective ∴ $dt \mapsto z - z'$ ∴ $[z] = [z']$

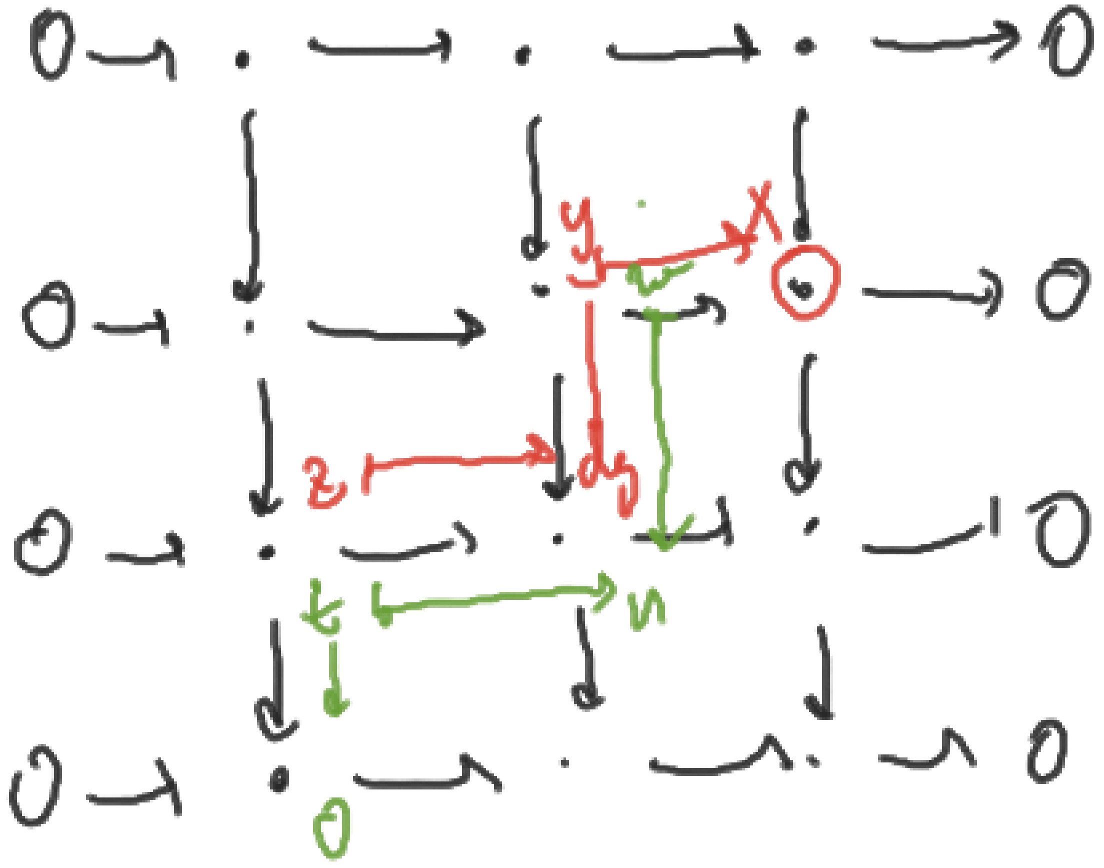
"follow your nose"

"diagram chase"

We still need to prove exactness.

$$\to H_nC \xrightarrow{i_*} H_nD \xrightarrow{j_*} H_nE \xrightarrow{\partial} H_{n-1}C \xrightarrow{i_*} H_{n-1}D \xrightarrow{j_*} H_{n-1}E \to$$

? $\operatorname{Ker} i_* = \operatorname{Im} \partial$



$\underbrace{i_* \partial}_{} x = 0$

$[i(z)] = [dy] = 0$

? $dt = 0$ $i(t) = u$ $[u] = 0$

$u = dw$

Put $x := j(w)$ $\left(dx = djw = jdw = ji(u) = 0 \right)$

By definition,

$[t] = \partial [x]$.

✓