MATH 592 3/11/2024

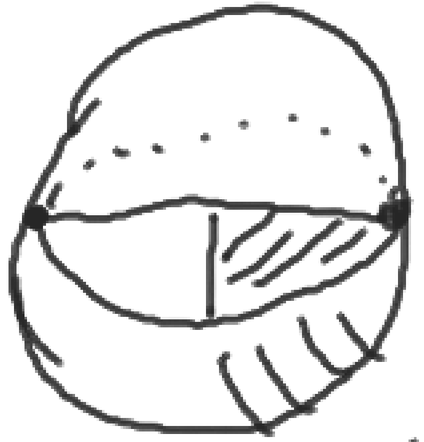
Exam Wednesday 3/13 in class.

Bring Colored Pen(cil)s !

More review examples in discussion tomorrow!

Review: (1) Describe the universal covering of the CW-complex X, Y.

(a)  $X = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\} \cup \{(x,0,0) \mid -1 \leq x \leq 1\}$

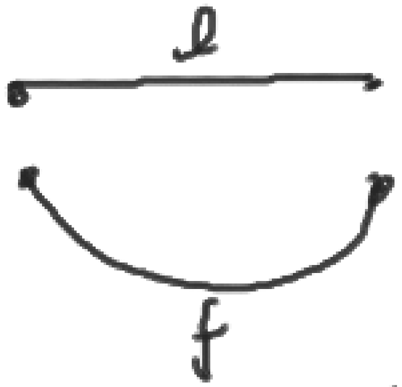
(b)  $Y = X \cup \{(r\cos\alpha, r\sin\alpha, 0) \mid 0 \leq r \leq 1, \frac{3\pi}{2} \leq \alpha \leq 2\pi\}$

(subspace topology of $\mathbb{R}^3$).

ⓐ CW-structure on X:

two 0-cells $(-1,0,0)$, $(1,0,0)$

two 1-cells



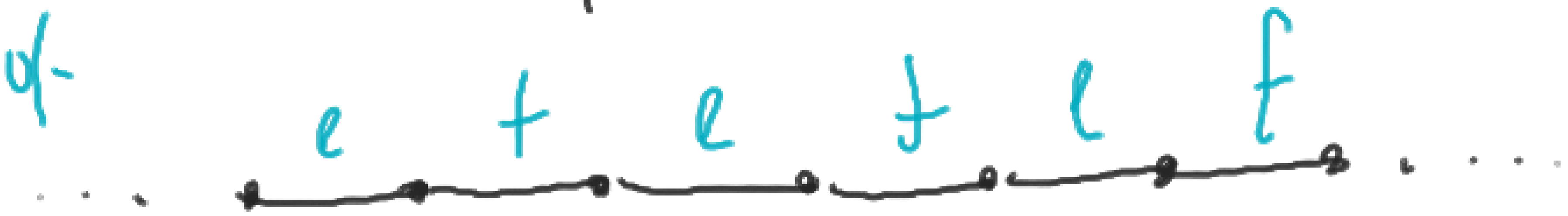
one 2-cell

g ← boundary attaches trivially.

$\pi_1 X = \mathbb{Z}$

Universal covering of $X_1 \cong S^1$: $\mathbb{R}$

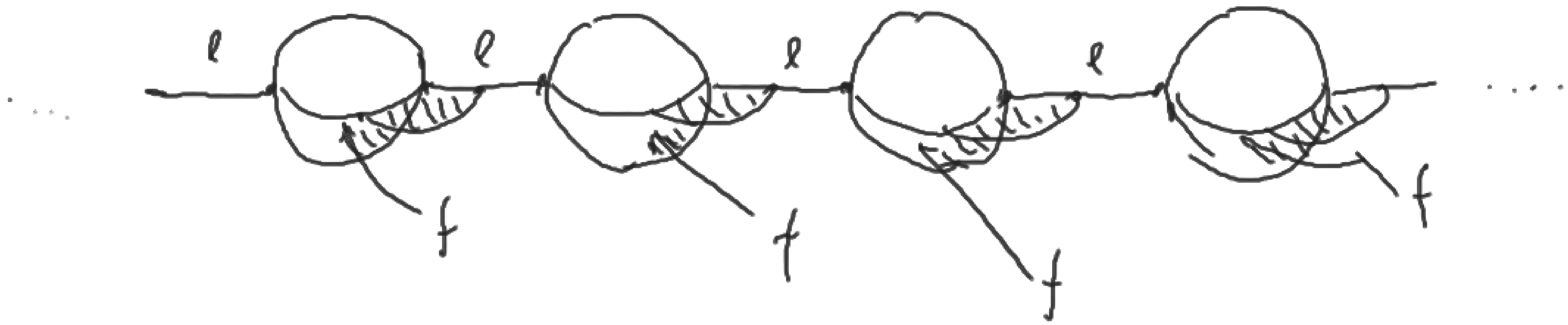
lifts of



Attach all possible lifts of the 2-cell:

(b) $X \hookrightarrow Y$ induces an $\cong$ on π_1

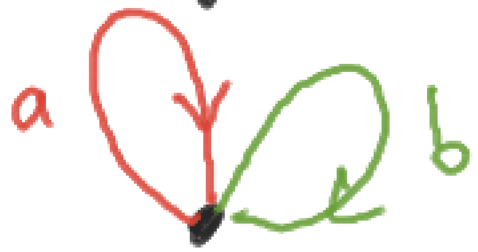
Universal coverings $\Rightarrow$ $\tilde{X} \hookrightarrow \tilde{Y}$



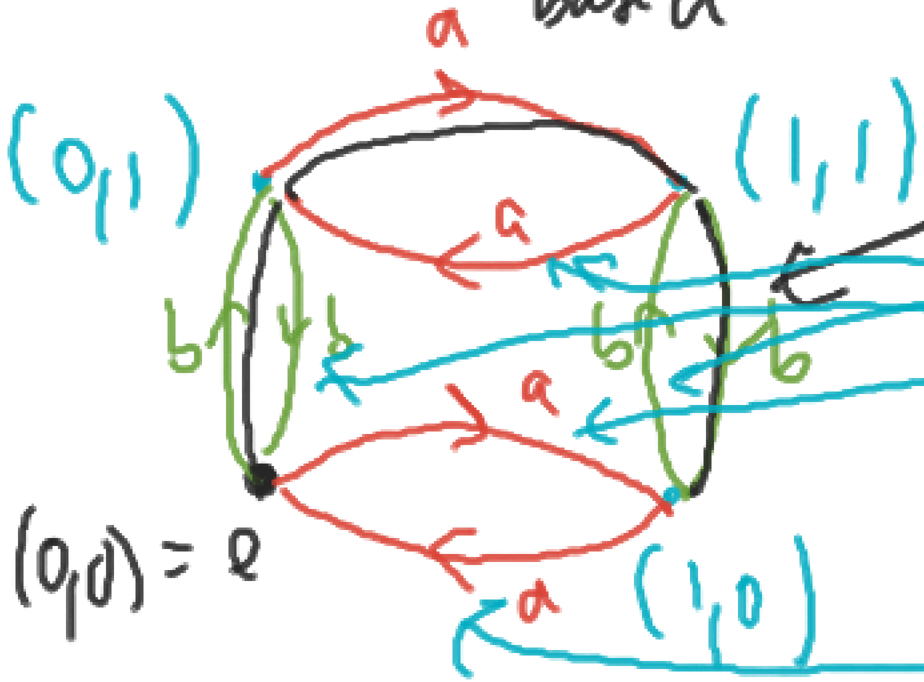
② Let a homomorphism $h : F(a,b) \longrightarrow (\mathbb{Z}/2 \times \mathbb{Z}/2, +)$ ($F(a,b)$ = free group on a, b) be given by $h(a) = (1,0)$, $h(b) = (0,1)$.
Find the free generators and the rank of $\operatorname{Ker}(h)$.

<u>Solution</u>: Construct the Cayley graph of $(\mathbb{Z}/2 \times \mathbb{Z}/2, +)$.

= the based covering space of



corresponding to the subgroup $\operatorname{Ker} h$.



spanning tree

rank = # edges not in spanning tree $= \underline{\underline{5}}$

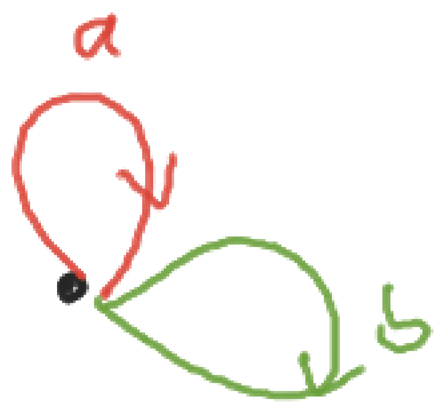
$\underbrace{b^2,\ ba^2b^{-1},\ bab^2a^{-1}b^{-1},\ ab^{-1}a^{-1}b^{-1},\ baba}$

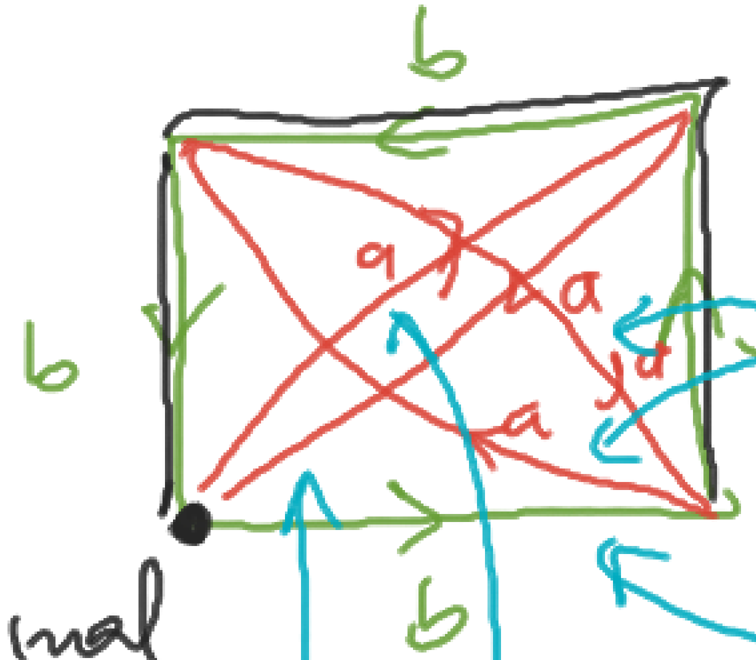
other answers possible

(3) Find the rank and free generators of the subgroup of the free group $F(a,b)$ on a,b spanned by

$a^{-1}b^{-2}, b^4, ab^2, bab, ba^{-1}b, b^3ab^3$. Is the subgroup normal?

<u>Solution</u>: Construct the covering of



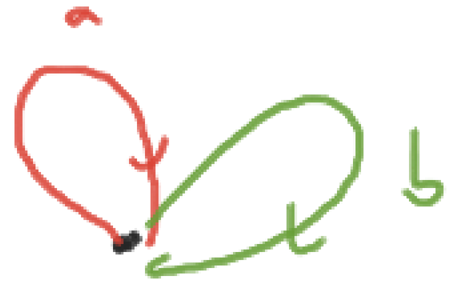


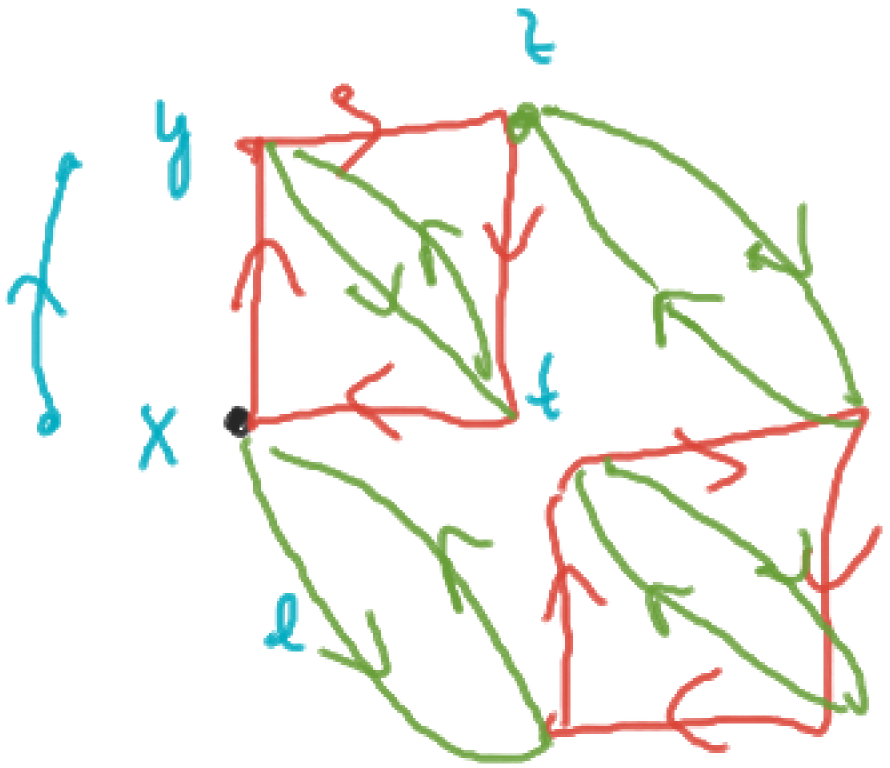
rank = 5

Yes, regular $\Rightarrow$ subgroup normal

($\exists$ unbased automorphism carrying the base point to every other vertex)

$b^{-2}a, \; ab^2, \; b^{-1}ab^3, \; b^{-3}ab, \; b^4$ (one answer)

Example: Is this covering of  regular?



not regular.

$\nexists$ automorphism sending $x \mapsto y$

because it would have to preserve the red square $xyzt$, but the edge e leaves the square.

④ Something on the equivalence between the unbased covering category of a connected CW-cx. X and the category $\pi_1(X,x)$-Set (or based connected coverings and subgroups of $(\pi_1(X,x), \subseteq)$).

Several possible types of problems

<u>Example</u>: Describe the orbit category of $\mathbb{Z}/2 \times \mathbb{Z}/2 = G$

orbit = transitive G-sets

$$\begin{tikzcd}
 & G/(\mathbb{Z}/2 \times \mathbb{Z}/2) & \\
G/\langle(1,0)\rangle \arrow[ur] \arrow[loop right, "\mathbb{Z}/2 \cong C"] & G/\langle(1,1)\rangle \arrow[u] \arrow[loop right, "\mathbb{Z}/2 \cong B"] & G/\langle(0,1)\rangle \arrow[ul] \arrow[loop right, "\mathbb{Z}/2 \cong A"] \\
 & G/\{(0,0)\} \arrow[ul] \arrow[u] \arrow[ur] \arrow[loop right, "\mathbb{Z}/2 \times \mathbb{Z}/2"] &
\end{tikzcd}$$

$A = G/\langle(0,1)\rangle$

$B = G/\langle(1,1)\rangle$

$C = G/\langle(1,0)\rangle$

Example: If a connected CW-cx. X has $\pi_1(X) = \mathbb{Z}/3 \times \mathbb{Z}/3$, how many non-isomorphic covering spaces does X have? (in the unbased category)

same as based if $\pi_1(X)$ abelian
otherwise, "unbased = based / conjugation"

? How many different subgroups does $\mathbb{Z}/3 \times \mathbb{Z}/3$ have?

$\mathbb{Z}/3 \times \mathbb{Z}/3$, $\{(0,0)\}$

$\mathbb{Z}/3$ - subgroups: generated by a non-zero element x / $\langle x \rangle = \langle -x \rangle$

$\langle(1,0)\rangle, \langle(0,1)\rangle, \langle(1,1)\rangle, \langle(1,2)\rangle$

$\underbrace{\underbrace{\text{non-zero element}}_{8} \; x / \langle x \rangle = \langle -x \rangle}_{4}$

Answer: $\underline{\underline{6}}$.