MATH 592 3/18/2024

Proof of the homotopy axiom for singular homology:

If we have a homotopy $h : f \simeq g : X \to Y$ (spaces) (more generally, pairs)

then we have a chain homotopy

$$k : Cf \simeq Cg$$

$$k_n : C_n X \longrightarrow C_{n+1} Y$$

It suffices to consider the homotopy

$$h_t : \Delta^n \longrightarrow \Delta^n \times [0,1]$$

$$h_t\,[s_0, \dots s_n] = ([s_0, \dots s_n], t)$$

$$f : \Delta^n \to \Delta^n \times [0,1]$$
$$s \longmapsto (s, 0)$$
$$g : s \longmapsto (s, 1)$$

We construct the value of the chain homotopy

$$k : C\Delta^n \longrightarrow C(\Delta^n \times [0,1])$$

on $\imath = \mathrm{Id} : \Delta^n \to \Delta^n$. The condition is that it actually be a chain homotopy

$$dk + kd = Cg - Cf. \quad \longleftarrow \text{ on } \imath : \Delta^n \to \Delta^n$$

Triangulation of the prism. We have particular singular $(n+1)$-simplices in $\Delta^n \times [0,1]$

$$\varphi_i : \Delta^{n+1} \longrightarrow \Delta^n \times [0,1] \qquad i = 0, \dots, n$$

$$[t_0, \dots t_{n+1}] \longmapsto \left([t_0, \dots, t_i + t_{i+1}, \dots t_{n+1}], \; t_{i+1} + \dots + t_{n+1} \right)$$

pairs that cancel : $\underbrace{\partial_{i+1}\varphi_i}$ vs. $\underbrace{\partial_{i+1}\varphi_{i+1}}$

$$k(\sigma) := \sum_{i=0}^{n} (-1)^i \varphi_i$$

$\partial_0 \varphi_0 \quad ([t_1, \dots t_{n+1}], 1)$

Where do $Cf(\sigma)$, $C(g(\sigma))$ come in?

$([t_0, \dots, t_n], 0) \quad ([t_0, \dots t_n], 1)$

$\partial_{n+1}\varphi_n \quad ([t_0, \dots t_n], 0)$

$(-1)^{n+1} \cdot (-1)^n = -1$

The general formula:

$$h : f \simeq g : X \longrightarrow Y$$

$$k_n : C_n X \longrightarrow C_{n+1} Y$$

$$\sigma : \Delta^n \longrightarrow X$$

$$\boxed{k_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, h \circ \varphi_i}$$

$$dk + kd = Cg - Cf$$

This proves the homotopy axiom. $\square$

Munkres: Elements of algebraic topology

$$\sum_{i=0}^{n} (-1)^i \varphi_i : \Delta^{n+1} \to \Delta^n \times [0,1]$$

because it is true here

Proof of the excision axiom:

$Z \subseteq Y \subseteq X$

$\mathrm{Interior}_X(Y) \supseteq \mathrm{Closure}_X(Z)$

Then $\subseteq_*: H_n(X \setminus Z, Y \setminus Z) \xrightarrow{\cong} H_n(X, Y)$ is an isomorphism.

Proposition: Let $\mathcal{U}$ be a set of subsets U of X, where $\bigcup_{U \in \mathcal{U}} \mathrm{Interior}_X(U) = X$.

Then let $C^{\mathcal{U}}(X)$ be the chain subcomplex of CX where $C^{\mathcal{U}}_n(X) = \mathbb{Z}\{\sigma : \Delta^n \to X \mid \exists U \in \mathcal{U} \quad \sigma(\Delta^n) \subseteq U\}$.

The inclusion

$$C^{\mathcal{U}}X \xrightarrow{\subseteq} CX$$

induses an isomorphism in homology.

Proof of excision using Proposition: $\mathcal{U} = \{Y, X \setminus Z\}$. We have a diagram of chain complexes

$$
\begin{tikzcd}
0 \arrow[r] & C(Y) \arrow[r] \arrow[d, "\mathrm{Id}"'] & C^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\subseteq"] & C(X \setminus Z, Y \setminus Z) \arrow[r] \arrow[d, "\subseteq"] & 0 \\
0 \arrow[r] & C(Y) \arrow[r] & C(X) \arrow[r] & C(X, Y) \arrow[r] & 0
\end{tikzcd}
$$

Induces $\cong$ in homology

Induces $\cong$ in homology by Proposition.

$\therefore$ induces $\cong$ in Homology by the long exact sequence and the 5-lemma

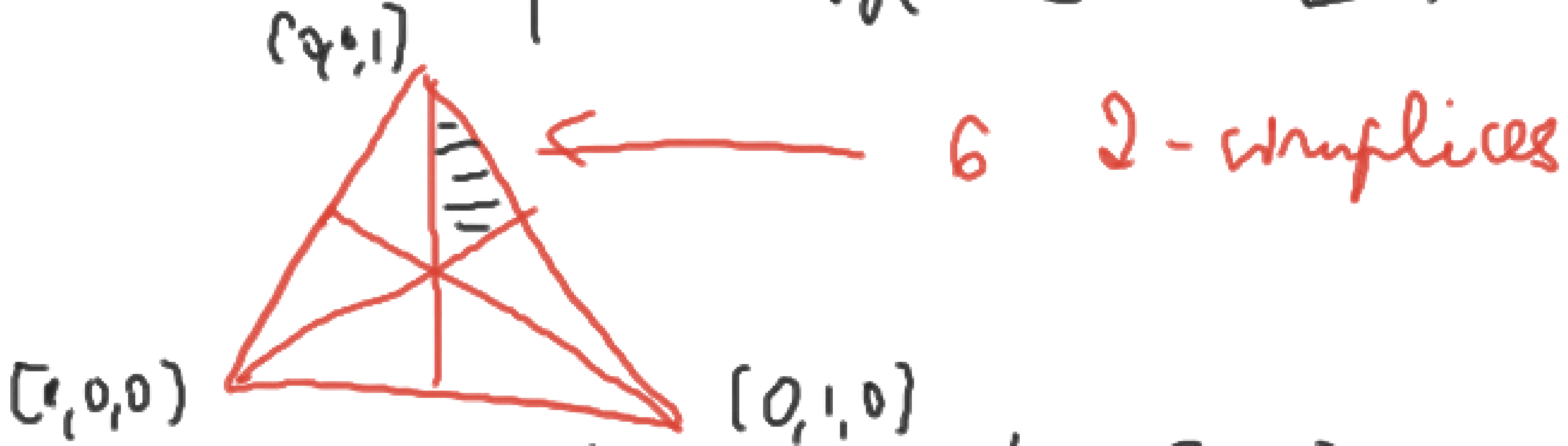
$\therefore \cong$

$$
\begin{tikzcd}
\arrow[r] & \bullet \arrow[r] \arrow[d, "\cong"] & \bullet \arrow[r] \arrow[d, "\cong"] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d, "\cong"] & \bullet \arrow[r] \arrow[d, "\cong"] & {} \\
\arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & {}
\end{tikzcd}
$$

exact

$\square$

<u>Barycentric subdivision</u>: let $\alpha : \{0, \dots, n\} \longrightarrow \{0, \dots, n\}$ be a permutation. We will define $\lambda_\alpha : \Delta^n \to \Delta^n$.



$$\left.\begin{aligned}
\lambda_{Id}[1, 0, \dots 0] &= \left[\tfrac{1}{n+1}, \tfrac{1}{n+1}, \dots \tfrac{1}{n+1}\right] \\
\lambda_{Id}[0, 1, 0, \dots 0] &= \left[0, \tfrac{1}{n}, \dots, \tfrac{1}{n}\right] \\
&\ \vdots \\
\lambda_{Id}[0, \dots, 0, 1] &= [0, \dots 0, 1]
\end{aligned}\right\} \text{ extend in an affine way}$$

$$\lambda_\alpha[t_0, \dots, t_n] = \lambda_{Id}[t_{\sigma^{-1}(0)}, \dots, t_{\sigma^{-1}(n)}].$$

$$sd(\sigma) := \sum_\alpha \mathrm{sign}(\alpha)\, \sigma \circ \lambda_\alpha$$

$+$ if α even
$-$ if α odd.

One verifies that this is a chain map.

<u>Lemma</u>: There exists a natural chain homotopy

$h : sd \simeq Id$.

$f : X \to Y$

$$\begin{tikzcd} C_n X \arrow[r, "h_n"] \arrow[d, "C_n f"'] & C_{n+1} X \arrow[d, "C_{n+1} f"] \\ C_n Y \arrow[r, "h_n"] & C_{n+1} Y \end{tikzcd}$$

A brief intermezzo

reduced homology

HW 1. Define, for a based space X, $\tilde{H}_n X := H_n(X, *)$.

(a) Prove a long exact sequence for a pair (X, Y), $* \in Y$

$$\to \tilde{H}_n(Y) \to \tilde{H}_n(X) \to H_n(X, Y) \to \tilde{H}_{n-1}(Y) \to \tilde{H}_{n-1}(X) \to \cdots$$

(b) Prove that $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$, $H_n(X) \cong \tilde{H}_n X$ for $n > 0$
$\forall X$ based

[Hint: The base point is always a retract $* \to X \to *$.]