MATH 592 3/20/2024

Lemma: There exists a natural homotopy

$$h : sd \simeq Id$$

Proof: $sd_0 : C_0X \to C_0X$ $= Id_{C_0X}$ $\boxed{dh + hd = Id - sd}$ ⊛

$h_{-1} := 0$, $h_0 := 0$

Suppose h_k constructed for $k < n$ satisfying ⊛ when applicable. Constructing h_n:

It is necessary and sufficient to give $\boxed{h_n(\imath_n)} \in C_{n+1}(\Delta^n)$

$\imath_n : \Delta^n \xrightarrow{Id} \Delta^n$ $(X = \Delta^n)$ $\sigma : \Delta^n \to X \in S_n X$

Taking $f = \sigma$ forces $h_n(\sigma) := \sigma \circ h_n(\imath_n)$.

$sd : C? \to C?$

natural: $f : X \to Y$

$$\begin{tikzcd} CX \arrow[r, "sd"] \arrow[d, "Cf"'] & CX \arrow[d, "Cf"] \\ CY \arrow[r, "sd"] & CY \end{tikzcd}$$

$$\boxed{\begin{tikzcd} C_nX \arrow[r, "h_n"] \arrow[d, "C_nf"'] & C_{n+1}X \arrow[d, "C_{n+1}f"] \\ C_nY \arrow[r, "h_n"] & C_{n+1}Y \end{tikzcd}}$$

$\mathbb{Z}S_nX = C_nX$

$? \quad h(z_n) \in C_{n+1}(\Delta^n) \qquad z_n = \mathrm{Id} : \Delta^n \to \Delta^n$

$? \quad dh(z_n) + \underbrace{hd(z_n)}_{\text{known by induction hypothesis}} = z_n - sd(z_n)$

$$dh(z_n) = \underbrace{z_n - sd(z_n) - hd(z_n)}_{?\text{ boundary ? if yes, we are done}} \in C_n\Delta^n \Big\} \text{ equation we are solving}_{n \geq 1}$$

Is this element a cycle?

$$d(z_n - sd(z_n) - hd(z_n)) = \underbrace{dz_n - dsd(z_n)}_{\underbrace{hddz_n}_{0} + dhdz_n} - dhd(z_n) = 0$$

one symbol ; $\Leftarrow$ induction hypothesis

$\therefore z_n - sd(z_n) - hd(z_n) = dy \quad \exists y \in C_{n+1}\Delta^n \quad$ because $H_n\Delta^n = 0$, $\Big|^{n \geq 1}_{\Delta^n \simeq *}$

Set $h(z_n) := y$. $\square$

Note: This holds with sd replaced by any additive natural transformation $s : C_?^{} \to C_?^{}$ where $s_0 = \mathrm{Id}$.

Recall the Proposition: If $\mathcal{U}$ is a set of subsets of X such that $\bigcup_{U \in \mathcal{U}} \mathrm{Interior}(U) = X$ then $C_{\mathcal{U}} X \xrightarrow{\subseteq} CX$ induces an isomorphism in homology.

$C_{\mathcal{U}} X_n = \mathbb{Z}\{\sigma : \Delta^n \to X \mid (\exists U \in \mathcal{U})\ \sigma(\Delta^n) \subseteq U\}$

Proof of the Proposition: Consider the short exact sequence

$$0 \to C_{\mathcal{U}}(X) \to CX \to CX/C_{\mathcal{U}}(X) \to 0$$

By the long exact sequence in homology, it suffices to prove that

$$H_n(CX/C_{\mathcal{U}}X) = 0 \quad \text{for all } n \in \mathbb{Z}. \qquad (\#)$$

A cycle in $(CX/C_{\mathcal{U}}X)_n$ is represented by $c \in C_n X$, $\boxed{dc \in (C_{\mathcal{U}}X)_{n-1}}$.

By the Lebesgue number theorem, there exists an $N \in \mathbb{N}$ such that $sd^N(c) \in C^{\mathcal{U}}(X)$. We have $k: sd^N \simeq Id$.

$$dk(c) + kd(c) = sd^N(c) - c$$

$$c + dk(c) = sd^N(c) - kd(c) \in C_{\mathcal{U}}X$$

By naturality, $kd(c) \in (C_{\mathcal{U}}X)_n$

$sd^N(c) \in (C_{\mathcal{U}}X)_n$

$c \in \mathrm{Im}\, \partial + C^n X$

Therefore, $[c] = 0 \in H_n(CX / C^n X)$. $\square$

Now we have proved the Eilenberg-Steenrod axioms, so we want to use them to compute the homology of a CW-complex.

---

n-sphere

<u>Theorem</u>: $\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$

$n > 0$ $\quad H_k(S^n) = \mathbb{Z}$ $k = 0, n$; 0 else

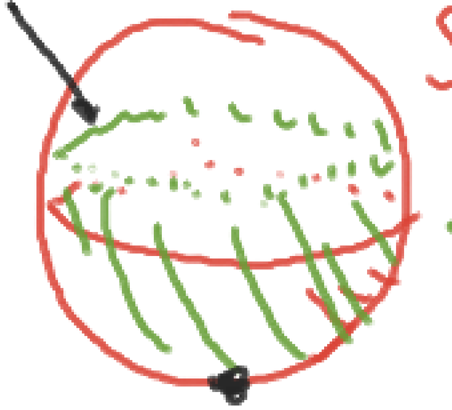
$n = 0$ $\quad H_k(S^0) = \mathbb{Z} \oplus \mathbb{Z}$ $k = 0$; 0 else

$H_0(S^0) = \mathbb{Z}$ $\quad$ $H_k(S^0) = 0$

Proof: Induction on n, $k > 0$

$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$

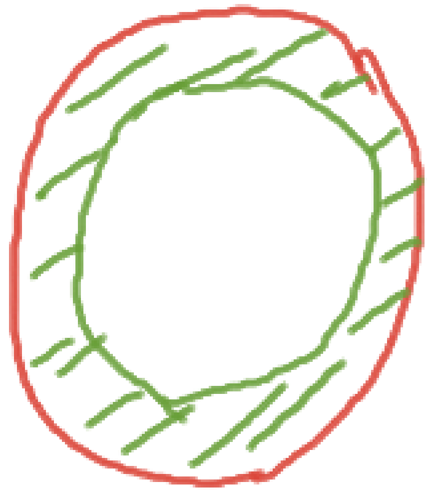
$S^n_+ = \{x \in S^n \mid x_n \geq 0\}$ $\quad$ $S^n_{--} = \{x \in S^n \mid x_n < 0\}$



$S^n_- = \{x \in S^n \mid x_n \leq 0.23\}$ $\quad$ $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$

$* = (0, \dots, 0, -1)$ (for now)

$$\tilde{H}_k(S^n) = H_k(S^n, *) \overset{\text{homotopy axiom}}{=} H_k(S^n, S^n_-) = H_k(\underbrace{S^n \setminus S^n_{--}, S^n_- \setminus S^n_{--}}_{\simeq (D^n, S^{n-1})}) =$$



$$= H_k(D^n, S^{n-1}).$$

Pick a base point $* \in S^{n-1}$. Long exact sequence in reduced homology for the pair (D^n, S^{n-1}) | We proved $\tilde{H}_k(S^n) = H_k(D^n, S^{n-1})$

$\tilde{H}_k(*) = 0$
$\forall k \in \mathbb{Z}$

$$\begin{array}{ccccccc} \tilde{H}_k(D^n) & \to & H_k(D^n, S^{n-1}) & \xrightarrow{\partial} & \tilde{H}_{k-1}(S^{n-1}) & \to & \tilde{H}_{k-1}(D^n) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

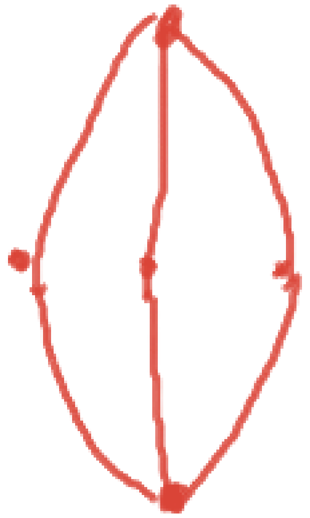
$\therefore \cong$ (by LES)

$\therefore \tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1})$. The induction is complete. $\square$

$H_0 X = \mathbb{Z}\pi_0 X$ ← Set of path-components

HW ② The unreduced suspension SX of a space X is defined as

$$SX = X \times [0,1] \Big/ \begin{array}{l}(x,0)\sim(y,0)\\ (x,1)\sim(y,1)\end{array}, \quad x,y\in X$$



$= S(\dots)$

Prove that if X is a based space then

$$\tilde{H}_k SX \cong \tilde{H}_{k-1} X \quad \text{for all } k \in \mathbb{Z}$$

(Put the base point of SX at $(x,0)$.)

In algebraic topology, this phenomenon is called stability.

HW 3 Prove that for every path-connected space X, $\pi_1(SX)=0$.