MATH 592 3/22/2024

Homology of a CW-complex X

Notation: Set of n-cells $= I_n$

Proposition: $H_k(X_n, X_{n-1}) = \begin{cases} \mathbb{Z} I_n & \text{when } k = n \\ 0 & \text{else} \end{cases}$

Motivation: $H_k(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$

Proof: $D^n = \{(x_1, \dots x_n) \in \mathbb{R}^n \mid \sum x_k^2 \leq 1\}$.

$D_0^n := D^n \setminus \{(0, \dots, 0)\}$

$E^n := \{(x_1, \dots x_n) \in D^n \mid \sum x_k^2 \geq \frac{1}{2}\}$

Recall: X_n is a pushout

attaching map

$$\begin{tikzcd} S^{n-1} \times I_n \arrow[r, "f_n"] \arrow[d, "\subseteq"'] & X_{n-1} \arrow[d] \\ D^n \times I_n \arrow[r] & X_n \end{tikzcd}$$

Define X_n^0 as a pushout

$$\begin{tikzcd} S^{n-1} \times I_n \arrow[r, "f_n"] \arrow[d] & X_{n-1} \arrow[d] \\ D_0^n \times I_n \arrow[r] & X_n^0 \end{tikzcd}$$

deformation retract $\therefore$ homotopy eq.

$$(X_n, X_n^0) \simeq (X_n, X_{n-1}).$$

$\therefore$

$$H_k(X_n, X_{n-1}) = H_k(X_n, X_n^0)$$

$$E = \coprod_{I_n} E^n \subseteq X_n$$

$Y := X_n^0$

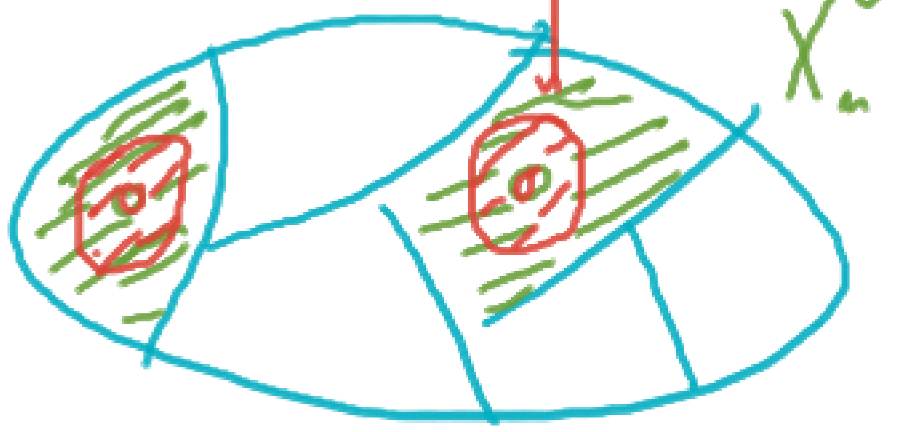
$\mathrm{Closure}_{X_n}(Z) \subseteq \mathrm{Interior}_{X_n}(Y)$

$Z := X_n \setminus E$

$$H_k(X_n, X_{n-1}) = H_k(X_n, X_n^0) = H_k\Big(E, \underbrace{E \cap X_n^0}_{\coprod_{I_n} E^n \setminus \{(0,\dots,0)\}}\Big)$$

$E \cap X_n^0$

$n=2$



X_n^0

X_{n-1}

X_n has 2 2-cells

$$(E, E\cap X_n^0) = \coprod_{I_n} \big(\underbrace{E^n, E^n \setminus \{(0,\dots,0)\}}_{\simeq (D^n, S^{n-1})}\big)$$

$$= \bigoplus_{I_n} H_k(D^n, S^{n-1})$$

$\mathbb{Z}$ when $k = n$

0 else.

$\square$

$$X_{n-2} \subseteq X_{n-1} \subseteq X_n$$

$$0 \to C(X_{n-1}, X_{n-2}) \to C(X_n, X_{n-2}) \to C(X_n, X_{n-1}) \to 0$$

Connecting map of long exact sequence in homology:

$$d := \partial : H_n(X_n, X_{n-1}) \longrightarrow H_{n-1}(X_{n-1}, X_{n-2})$$

$$d : \mathbb{Z} I_n \to \mathbb{Z} I_{n-1}.$$

<u>Lemma</u>: $dd = 0$.

Proof: By naturality,

$d = \partial$ is the composition $H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \xrightarrow{\varepsilon_*} H_{n-1}(X_{n-1}, X_{n-2})$

two consecutive maps in LES for (X_{n-1}, X_{n-2}) ∴ composition $= 0$.

$$H_{n-2}(X_{n-2}, X_{n-3}) \xleftarrow{\varepsilon_*} H_{n-2}(X_{n-2}) \xleftarrow{\partial} H_{n-1}(X_{n-1}, X_{n-2})$$

In red: same thing
n replaced by n-1

□

Let X be a CW-complex. Define its <u>cell chain complex</u>

$$C^{cell}X: \qquad \cdots\cdots \to \mathbb{Z}I_n \xrightarrow{d} \mathbb{Z}I_{n-1} \longrightarrow \cdots\cdots$$

defined above

and its <u>cell homology</u>:

$$H^{cell}_n X := H_n C^{cell} X.$$

A cell map $f: X \to Y$, X, Y CW-complexes, satisfies $f(X_n) \subseteq Y_n$.

natural with respect to cell maps

<u>Theorem</u>: If X is a CW-complex, then we have an isomorphism

$$H^{cell}_n X \cong H_n X$$

singular homology

To prove the theorem, we state some lemmas:

<u>Lemma</u>: For $m \leq n$, $H_k(X_n, X_m) = 0$ if $k \leq m$ or $k > n$.

Proof: Induction on $n-m$. ($n-m=1 \leftarrow$ Proposition). For induction step, use long exact sequence of $X_m \subseteq X_{m+1} \subseteq X_n$. $\square$

$\therefore m=-1$: $H_k(X_n) = 0 \quad k > n$

$\therefore$ colimit axiom: $H_k(X, X_m) = 0 \quad k \leq m$.

? $H_n X_n$. Use a long exact sequence.

long exact sequence of the pair (X_n, X_{n-1})

$$H_n(X_{n-1}) \longrightarrow H_n(X_n) \longrightarrow H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1})$$

$H_n(X_{n-1}) = 0$ by lemma

injective

$$\therefore \quad H_n(X_n) = \operatorname{Ker}\left(\partial : H_n(X_n, X_{n-1}) \longrightarrow H_{n-1}(X_{n-1})\right).$$

$H_n(X_n, X_{n-1}) = \mathbb{Z} I_n$

$$H_{n-1}(X_{n-1}) \xrightarrow{i_*} H_{n-1}(X_{n-1}, X_{n-2})$$

injective

$$= \operatorname{Ker}\left(\partial : H_n(X_n, X_{n-1}) \longrightarrow H_{n-1}(X_{n-1}, X_{n-2})\right)$$

$$\mathbb{Z} I_n \xrightarrow{d} \mathbb{Z} I_{n-1}$$

$\therefore \quad H_n(X_n) = \underbrace{n\text{-cycles of } C^{cell}X}_{Z_n^{cell}X}$

Now consider the long exact sequence of the pair (X, X_n)

$$H_{n+1}(X, X_n) \xrightarrow{\partial} H_n(X_n) \to H_n(X) \to \underset{\substack{\parallel \\ 0 \text{ by lemma}}}{H_n(X, X_n)}$$

$\uparrow$ we would like $H_{n+1}(X_{n+1}, X_n)$ instead.

Consider the LES of $X_n \subseteq X_{n+1} \subseteq X$.

$$H_{n+1}(X_{n+1}, X_n) \longrightarrow H_{n+1}(X, X_n) \longrightarrow \overset{\substack{0 \\ \parallel}}{H_{n+1}(X, X_{n+1})}$$

$\therefore$ onto

$\therefore$ By naturality, $\mathrm{Im}\left(\partial : H_{n+1}(X_{n+1}, X_n) \to H_n(X_n)\right) = \mathrm{Im}\left(\partial : H_{n+1}(X, X_n) \to H_n(X_n)\right)$ $\square$

We proved the Theorem. The question remains how to calculate

$d : \mathbb{Z} I_n \to \mathbb{Z} I_{n-1}$.

$$\begin{array}{cc} & \quad e \\ I_{n-1} \left\{ f \right. & \underbrace{\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}}_{I_n} \end{array}$$

matrix with entries in $\mathbb{Z}$

columns only have finitely many non-zero entries.

$e \in I_n$ has an attaching map

$$\varphi_e : S^{n-1} \to X_{n-1}$$

$$X_{n-1} \xrightarrow[\text{the boundary of } f \text{ and everything else.}]{\text{contracts}} D^{n-1}/S^{n-2} \cong S^{n-1}$$

$\therefore$ φ_e specifies a map $S^{n-1} \longrightarrow S^{n-1}$. ? What does this map induce in $H_{n-1}(S^{n-1})$

$\uparrow$

given f

$$H_{n-1}(S^{n-1}) = H_n(D^n, S^{n-1})$$

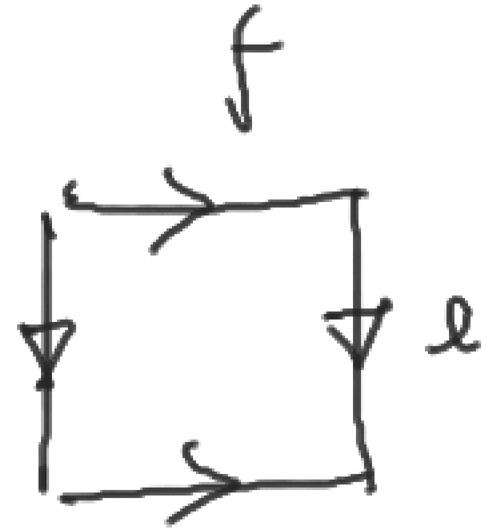
<u>Example</u>: $n = 2$. The attaching map of a 2-cell corresponds to a word in $\pi_1 X_1$ (up to conjugation).
The same linear combination of 1-cells is d_1^{cell}.

$\therefore \quad H_1 X = \pi_1(X)^{ab}$.

Example: ? $H_* T$

homology

all groups



connected

$$C^{cell} T \qquad \mathbb{Z} \longrightarrow \underset{e,f}{\mathbb{Z}^2} \xrightarrow{\ 0\ } \mathbb{Z}$$

$$(efe^{-1}f^{-1})^{ab} = e+f-e-f=0$$

$H_0 T = \mathbb{Z}$

$H_1 T = \mathbb{Z} \oplus \mathbb{Z}$

$\boxed{H_2 T = \mathbb{Z}}$

HW 4

a. In a similar way, calculate $H_* \underbrace{T \# \cdots \# T}_{n}$

b. In a similar way, calculate $H_* \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n}$.