MATH 592 3/25/2024

Calculating CW-homology - discussion tomorrow on more examples

We understand CW-complexes of dim. 2 : $X = X_2$ $\{n\text{-cells}\} = I_n$

$n = 0,1,2$

$$\mathbb{Z} I_2 \xrightarrow{d_2} \mathbb{Z} I_1 \xrightarrow{d_1} \mathbb{Z} I_0$$

2-cell α $\hat{=}$ conjugacy class $w \in \pi_1 X_1$

$d_2(\alpha) = w^{ab}$

$e \longmapsto$ beginning point (S) - end point (T)

We want to think of X_1 as a oriented graph

More precisely, the boundary of α determines the edges along it $\rightsquigarrow$ word in 1-cells.

<u>Example</u>: Let X be a 2-dimensional CW-complex where X_1:

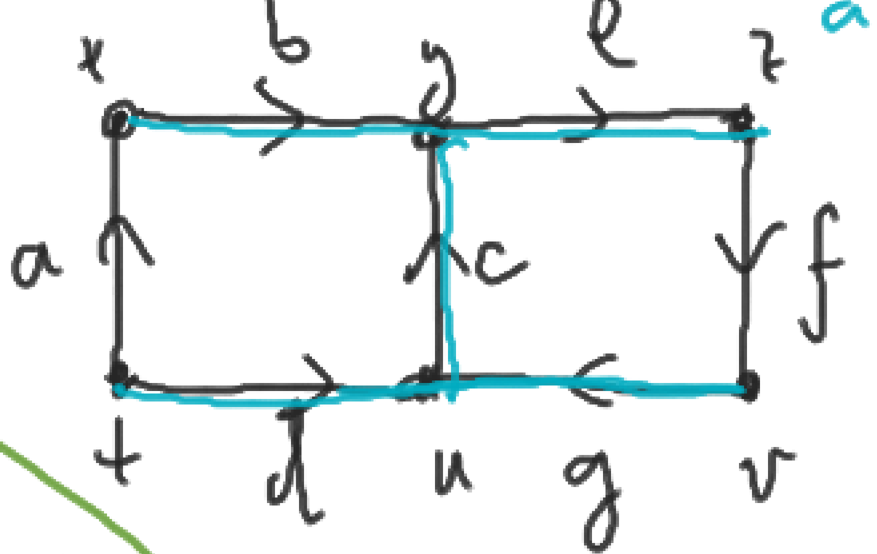
$\alpha \mapsto a-f$

$\beta \mapsto 4a+4f$

$\mathbb{Z}\{\alpha,\beta\} \to \mathbb{Z}\{a,f\} \xrightarrow{0} \mathbb{Z}$

2

Contract a spanning tree to a point (same homology by LES)



$a \mapsto t-x$, $b \mapsto x-y$, $c \mapsto u-y$, $d \mapsto t-u$

$e \mapsto y-z$, $f \mapsto z-v$, $g \mapsto v-u$

cycle in the graph X_1

There are two 2-cells α, β attached along α: $b c^{-1} g^{-1} f^{-1} e^{-1} c^{-1} d^{-1} a$

$\alpha \mapsto a+b-2c-d-e+f-g$

$\beta \mapsto 4a+4b+4e+4f+4g-4d$

β: $(a b e f g d^{-1})^4$

Calculate $H_* X$: $\mathbb{Z}\{\alpha,\beta\} \longrightarrow \mathbb{Z}\{a,b,c,d,e,f,g\} \longrightarrow \mathbb{Z}\{x,y,z,t,u,v\}$

$$
\begin{tikzcd}
 & & \mathbb{Z}/8 & \\
0 \arrow[r] & \mathbb{Z}\{\alpha,\beta\} \arrow[r] & \mathbb{Z}\{a,f\} \arrow[r, "0"] & \mathbb{Z}
\end{tikzcd}
$$

$d\alpha = a - f$

$d\beta = 4a + 4f$

$$
\begin{array}{cc} a & f \end{array}
$$

$$
\begin{pmatrix} 1 & -1 \\ 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 4 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}
$$

1

$\mathbb{Z}/8$

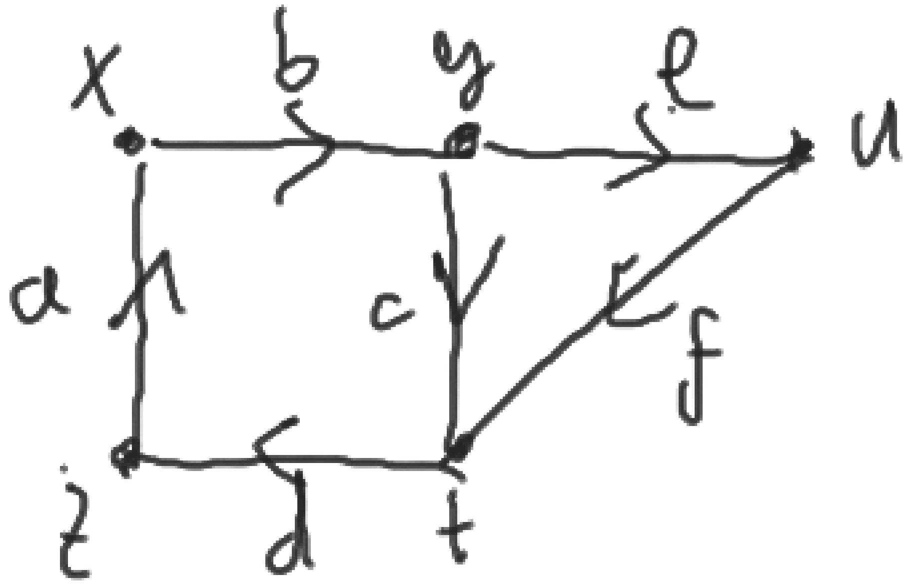
rank 2

$2-2$

$H_2 X = 0 \qquad H_1 X = \mathbb{Z}/8 \qquad H_0 X = \mathbb{Z}$

HW (1) Let $X = X_2$ be a CW-complex where

X_1:



There are 3 2-cells: α, β, γ, attached along the cycles

$\alpha : (abcd)^6$ $\quad \beta : (befda)^4$ $\quad \gamma : (efc^{-1})^{10}$.

Calculate $H_* X$.

Example: $T = S^1 \times S^1$

universal cover

$H_2 T = \mathbb{Z}$

$\pi_2 T = 0 \quad \left(= \pi_2 \tilde{T} \right)$

$\mathbb{R}^2 \simeq *$

Example: $X = S^2 \vee S^1$

1 2-cell

1 1-cell

1 0-cell

$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$

$H_2 X = \mathbb{Z}$

universal cover

$\pi_2 X = \pi_2 \tilde{X} = \bigoplus_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}[\mathbb{Z}]$.

$\tilde{X} = \bigvee_{\mathbb{Z}} S^2$

What happens in higher dimension? (CW X, $I_n = \{n\text{-cells}\}$

$$\mathbb{Z}[I_n] \xrightarrow{d} \mathbb{Z}[I_{n-1}]$$

$$\begin{array}{c} \\ e \in I_{n-1} \end{array} \underbrace{\left(\begin{array}{ccc} & & \\ & k_{e\alpha} & \\ & & \end{array} \right)}_{I_n} \Big\} I_{n-1}$$

$\alpha \in I_n$

finitely many non-zero entries in each column

Recipe from last time:

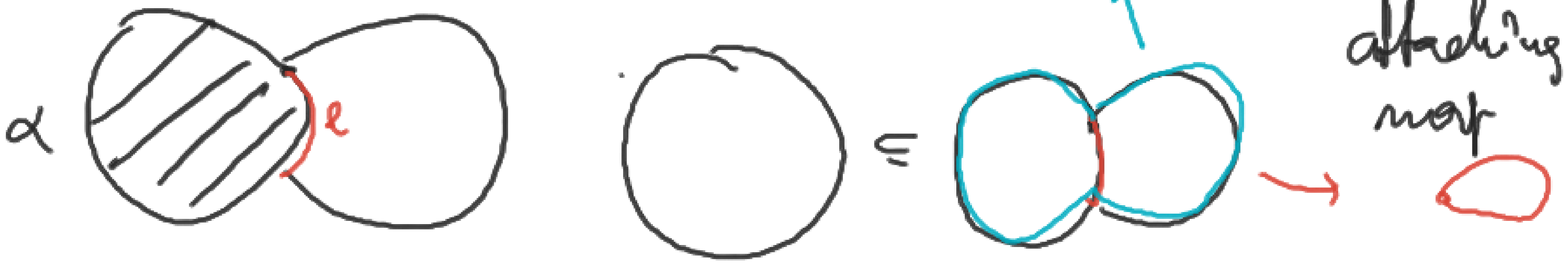
$$S^{n-1} \xrightarrow{f_\alpha} X_{n-1} \longrightarrow S^{n-1}$$

attaching map

$\cong$

e/its boundary

$= X_{n-1}/X_{n-2} \cup$ all $(n-1)$-cells except e



The coefficient of the matrix is whatever the above map

$$S^{n-1} \longrightarrow S^{n-1}$$

induces in H_{n-1}

$$\begin{tikzcd}
H_{n-1} S^{n-1} \arrow[r] \arrow[d, equal] & H_{n-1} S^{n-1} \arrow[d, equal] \\
\mathbb{Z} \arrow[r, "k \mapsto k \cdot \alpha"'] & \mathbb{Z}
\end{tikzcd}$$

the coefficient

$(m = n-1)$

<u>Question</u>: Given a map $f : S^m \longrightarrow S^m$, what does it induce of H_m?

Answer: $f : S^m \longrightarrow S^m$

$f \simeq$ smooth map

has a regular value: $x \in S^m$ $\quad f^{-1}(x) = \{y_1, \dots, y_\ell\}$

at each $i = 1, \dots, \ell$,

$$Df_{y_i} : TS^m_{y_i} \xrightarrow{\cong} TS^m_x$$

is an isomorphism.

orientation

$$\deg f := \sum_{i=1}^{\ell} \operatorname{sign} Df_{y_i}$$

$\operatorname{sign} Df_{y_i} = 1$ if Df_{y_i} preserves orientation

$\phantom{\operatorname{sign} Df_{y_i}} = -1$ if Df_{y_i} reverses orientation

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In basis of the given orientation: take $\operatorname{Det} Df_{y_i}$, take its sign.

Theorem: The map $f : S^m \to S^m$ induces on $H_m S^m = \mathbb{Z}$ multiplication by $\deg f$. (Proof: later)

Comment: non-smooth maps can have a regular value: f is smooth on some open neighborhood of $f^{-1}(x) = \{y_1, \dots, y_\ell\}$. Theorem still holds.

Example: $X = \mathbb{RP}^n = \{\text{lines in } \mathbb{R}^{n+1}\} = S^n / x \sim -x$.

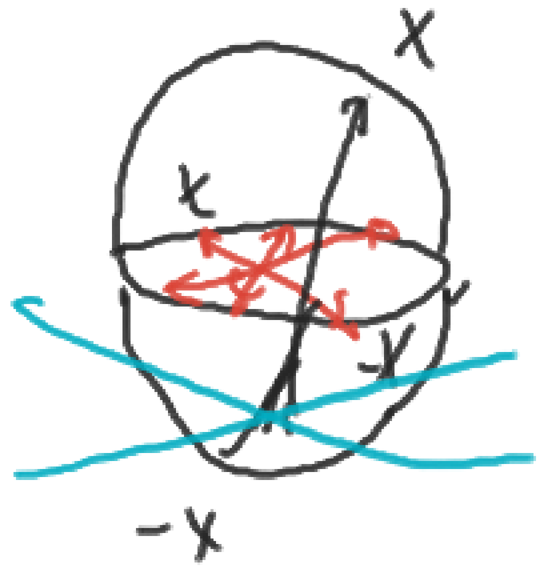
This is a CW-complex with 1 cell in every dimension $0, 1, 2, \dots, n$.

$X_i = \mathbb{RP}^i \qquad (S^i \subseteq S^n)$

$$(x_0, \dots, x_i) \longmapsto (x_0, \dots, x_i, 0, \dots, 0).$$

To construct $\mathbb{RP}^i$ from $\mathbb{RP}^{i-1}$,

$$\underbrace{D^i \cong \{x \in S^i \mid x_i \geq 0\}}_{S^{i-1}_+} \supseteq S^{i-1} \xrightarrow[\text{projection}]{} \mathbb{RP}^{i-1}$$



Compact → Hausdorff injective ⇒ CW axiom.

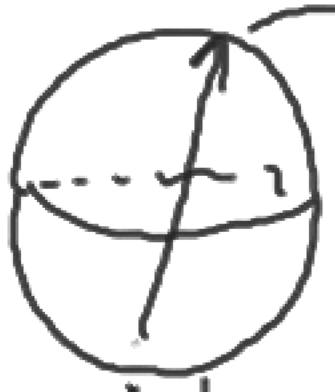
Calculating $H_* \mathbb{RP}^n$:

$$\underset{n}{\mathbb{Z}} \xrightarrow{d_n} \underset{n-1}{\mathbb{Z}} \xrightarrow{d_{n-1}} \mathbb{Z} \to \cdots \to \underset{1}{\mathbb{Z}} \xrightarrow{d_1} \underset{0}{\mathbb{Z}}$$

? d_i

Attaching map = projection

$$S^{i-1} \to S^{i-1} \cong S^{i-1}_+ / S^{i-2}$$



orientation +1

$x \mapsto -x$ and apply the same map.

$1 + (-1)^{\sigma(i)}$

$\sigma(i) =$ orientation of $x \longmapsto -x$ on S^{i-1}

sign reversal in $(i-1)$ dim

$\sigma(i) = (-1)^{i}$

1

more next time

$C^{cell}(\mathbb{RP}^n)$

$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \cdots \overset{2}{} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$