MATH 592 3/27/2024

The (cellular) homology of $\mathbb{RP}^n$
= singular

$$\mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \mathbb{Z} \quad \cdots\cdots \quad \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z}$$

? d_k We noted that $x \longmapsto -x : S^m \to S^m$
has degree $(-1)^{m-1}$.

← does orientation change? (Munkres: Analysis on manifolds)

Attaching map of the k-cell : $S^{k-1} \to S^{k-1}$

$\therefore\ d_k = 1 + (-1)^{(k-1)-1} = 1 + (-1)^k \qquad k \geq 2$

n even $\quad \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \cdots\cdots \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$

$H_* : \quad 0 \quad \mathbb{Z}/2 \quad 0 \qquad\qquad 0 \quad \mathbb{Z}/2 \quad \mathbb{Z}$

<u>n even</u> : $H_k \mathbb{RP}^n = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2 & 0<k<n \text{ odd} \\ 0 & \text{else.} \end{cases}$

<u>n odd</u> :

$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$

$\mathbb{Z} \quad 0 \quad \mathbb{Z}/2 \quad \cdot \quad \cdot \quad 0 \quad \mathbb{Z}/2 \quad \mathbb{Z}$

$H_k \mathbb{RP}^n = \begin{cases} \mathbb{Z} & k=0 \text{ or } n \text{ (odd)} \\ \mathbb{Z}/2 & 0<k<n \text{ odd} \\ 0 & \text{else.} \end{cases}$

<u>Note</u>: $\operatorname{colim}(\mathbb{R}P^1 \xrightarrow{\subset} \mathbb{R}P^2 \xrightarrow{\subset} \mathbb{R}P^3 \xrightarrow{\subset} \cdots \to \mathbb{R}P^n \to)$

$= \mathbb{R}P^\infty$.

$$H_k \mathbb{R}P^\infty = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2 & k>0 \text{ odd} \\ 0 & k>0 \text{ even.} \end{cases}$$

? The universal cover of $\mathbb{R}P^k$: S^k

$\therefore$ The universal cover of $\mathbb{R}P^\infty$: $S^\infty = \bigcup S^k$

$S^\infty \simeq *$.

S^k simply connected

$\mathbb{R}P^k = S^k / x \sim -x$

$\mathbb{Z}/2$ acts freely

$S^\infty = V(1)$ orthonormal frames

$V(n) = \{(v_1, \dots v_n) \in \mathbb{R}^\infty \mid \|v_i\| = 1 \quad v_i \cdot v_j = 0 \quad i \neq j\}$

free vector space on a countable basis $e_1, e_2, \dots$

$V(n) \overset{i}{\subseteq} W(n) = \{(v_1, \dots v_n) \in \mathbb{R}^\infty \mid v_i \text{ linearly independent}\}$

there exists a retraction $r: W(n) \to V(n)$ $\quad ir \simeq Id$.

Gram-Schmidt orthogonalisation process.

homotopy equivalence.

Claim: $W(n)$ is contractible (hence so is $V(n)$).

Proof: $\mathbb{R}^\infty \ni (a_1, a_2, \dots)$ $\exists N\ \forall n > N\ a_n = 0$.

We can apply linear homotopy.

$$\underbrace{t(a_1, a_2, \dots) + (1-t)(a_1, 0, a_2, 0, a_3, 0, \dots)}_{\text{never } 0.} \quad t \in [0,1]$$

$\neq 0$

linear injections

$\mathbb{R}^\infty \to \mathbb{R}^\infty$

φ

$(v_1, \dots v_n) \in V(n)$ compose id with this homotopy

$Id \simeq (a_1, a_2, \dots) \mapsto (a_1, 0, a_2, 0, \dots)$

similarly to $(0, a_1, 0, a_2, \dots)$

$\mathrm{Id} \simeq f : V(n) \longrightarrow V(n) \longleftarrow$ frames with only even non-zero coord

choose one frame $w = (w_1, \dots w_n)$ with only odd coord.

$\mathrm{cont}_w(V(n))$ linear homotopy f and w

$$t \cdot \underline{\qquad} + (1-t) \underline{\qquad} . \quad \square$$

G finite group. $G \subseteq O(n)$ $\nwarrow$ faithful real rep. (e.g. $\mathbb{R}G$ by left translation) G act on $g(h) := gh$.

$O(n)$ acts freely on $V(n)$ $\therefore$ G acts freely on $V(n)$,

∴ We have a covering map

$$\underset{\simeq *}{V(n)} \xrightarrow{\text{universal covering}} V(n)/G$$

One can prove that this is a CW-complex, unique up to homotopy equivalence: $\boxed{BG} =$

$= K(G,1)$

$\pi_1 K(G,1) = G$

$\pi_n K(G,1) = 0 \quad n>1$

(in fact, it exists for any discrete group)

$$\therefore B\mathbb{Z}/2 = K(\mathbb{Z}/2,1) \simeq \mathbb{R}P^\infty$$

$H_* BG =: H_* G$

$$H_* \mathbb{Z}/2 = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2 & n>0 \text{ odd} \\ 0 & \text{else.} \end{cases}$$

$$\Phi_n = \pi_1(\underbrace{T \# \cdots \# T}_{n}) = \langle a_1, b_1, \dots a_n, b_n \mid \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} \rangle$$

$\therefore$

$$T \# \cdots \# T = K(\Phi_n, 1) \qquad HW \Rightarrow \underbrace{H_2 \Phi_n = \mathbb{Z}}$$

$\therefore \Phi_n$ is not a free group. $\quad F_m :=$ free group on m elements

$$K(F_m, 1) = \bigvee_m S^1 \qquad \therefore H_1(F_m) = \mathbb{Z}^m \qquad \Big| \; H_1 G = G^{ab}$$

$$\underbrace{H_2(F_m) = 0}$$

Example: $\mathbb{C}P^n = \{\mathbb{C}\text{-lines in } \mathbb{C}^{n+1}\} = S^{2n+1}/S^1$ $\quad \lambda(z_0, \dots z_n) \sim (\lambda z_0, \dots, \lambda z_n)$

$[z_0, \dots z_n]$

$S^{2n+1} = \{(z_0, \dots z_n) \in \mathbb{C}^{n+1} \mid \sum |z_k|^2 = 1\}$

S^1 = unit sphere in $\mathbb{C}$

CW-decomposition:

$\mathbb{C}P^0 \subseteq \mathbb{C}P^1 \subseteq \dots \subseteq \mathbb{C}P^n$

$\mathbb{C}P^k \subseteq \mathbb{C}P^n$

$[z_0, \dots z_n] \longmapsto [z_0, \dots z_n, 0, \dots 0]$

How to make $\mathbb{C}P^{k+1}$ from $\mathbb{C}P^k$? Attach only one $2(k+1)$-cell:

$$\big((1-t)z_0, \dots (1-t)z_k, \underset{k+1}{t}, 0, \dots 0\big) \quad t \in [0,1]\big)$$

$t=0$

boundary

attaching map

$[z_0, \dots z_k] \in \mathbb{C}P^k$

$\sum_{i=0} |z_i|^2 = 1$

D^{2k+2}

boundary: $t=0$

$$\begin{tikzcd} S^{2k+1} \arrow[r] \arrow[d, hook] & \mathbb{C}P^k \arrow[d] \\ D^{2k+2} \arrow[r] & \mathbb{C}P^{k+1} \end{tikzcd}$$

bijective continuous, $\therefore \cong$

$H_* \mathbb{C}P^n$: $\quad \mathbb{Z} \xrightarrow{d} 0 \xrightarrow{d} \mathbb{Z} \xrightarrow{d} 0 \ \cdots\cdots \ \mathbb{Z} \xrightarrow{d} 0 \xrightarrow{d} \mathbb{Z}$

$2n \qquad 2n-2 \qquad\qquad 2 \qquad 0$

$$H_k \mathbb{C}P^n = \begin{cases} \mathbb{Z} & 0 \le k \le n \text{ even} \\ 0 & \text{else.} \end{cases}$$

(HW) (2) Calculate the homology of $\mathbb{RP}^n / \mathbb{RP}^k$ $\quad 0 \leq k \leq n$.

(3) Let $X = D^3/\sim$ (a) $(x_1, x_2, x_3) \sim (-x_1, x_2, x_3)$

$= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum x_i^2 \leq 1\}$ when $x_1^2 + x_2^2 + x_3^2 = 1$

(b) $(x_1, x_2, x_3) \sim (-x_1, -x_2, x_3)$

(4) Let $m \geq 2$. Find a CW-structure on $S^m \times S^m$ and use it to find $H_*(S^m \times S^m)$.

Next time: $B\mathbb{Z}/k$,
Proof of degree theorem

- Euler characteristic
- Mayer-Vietoris sequence