MATH 592 3/29/2024

<u>Example:</u> Let $S^{2n+1} = \{(z_0, \dots z_n) \in \mathbb{C}^{n+1} \mid \sum |z_n|^2 = 1\}$

$L_{2n+1}(k) = S^{2n+1} / (\mathbb{Z}/k)$ ← space of orbits

$\mathbb{Z}/k = \mu_k = \{\lambda \in \mathbb{C} \mid \lambda^k = 1\}$

Action: $\lambda \cdot (z_0, \dots, z_n) = (\lambda z_0, \dots \lambda z_n)$

$(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$

$S^{2n+1}/\sim \; = L_{2n+1}(k).$

CW-structure on $L_{2n+1}(k)$: "lens space"

We have one cell in each dimension $0, 1, 2, \dots, 2n+1$

$X = L_{2n+1}(k)$

$X_{2l+1} = L_{2l+1}(k)$

skeleton

Suppose X_{2l+1} is given a CW-structure. Attach a $(2l+2)$-cell

$$\left\{ (t z_0, \dots t z_l, (1-t), 0, \dots 0) \;\middle|\; 0 \le t \le 1,\ z_0, \dots z_l \in \mathbb{C},\ \sum_{j=0}^{l} |z_j|^2 = 1 \right\} \cong D^{2l+2}$$

Boundary: $S^{2l+1} \longrightarrow X_{2l+1} \left(= S^{2l+1} / \mathbb{Z}/k \right).$

Cell of dimension $(2l+3)$:

$$\left\{ (t z_0, \dots t z_l, (1-t), 0, \dots 0, s) \;\middle|\; 0 \le t \le 1, 0 \le s \le 1, z_0, \dots z_l \in \mathbb{C}, \sum_{j=0}^{l} |z_j|^2 = 1 \right\}$$

$$D^{2l+2} \times [0,1] \cong D^{2l+3}$$

Attaching: $t = 1$ attach

$$(z_0, \dots z_l, 0, \dots 0, s) = S^{2l+1} \times [0,1] \xrightarrow{\text{proj}} S^{2l+1} \longrightarrow S^{2l+1}/\mathbb{Z}/k = X_{2l+1}$$

$t = 0$ Attach

$(0, \dots 0, 1, 0 \dots 0, s)$ to $(0, \dots 0, 1, 0, \dots 0) \in (2l+2)$-dim cell

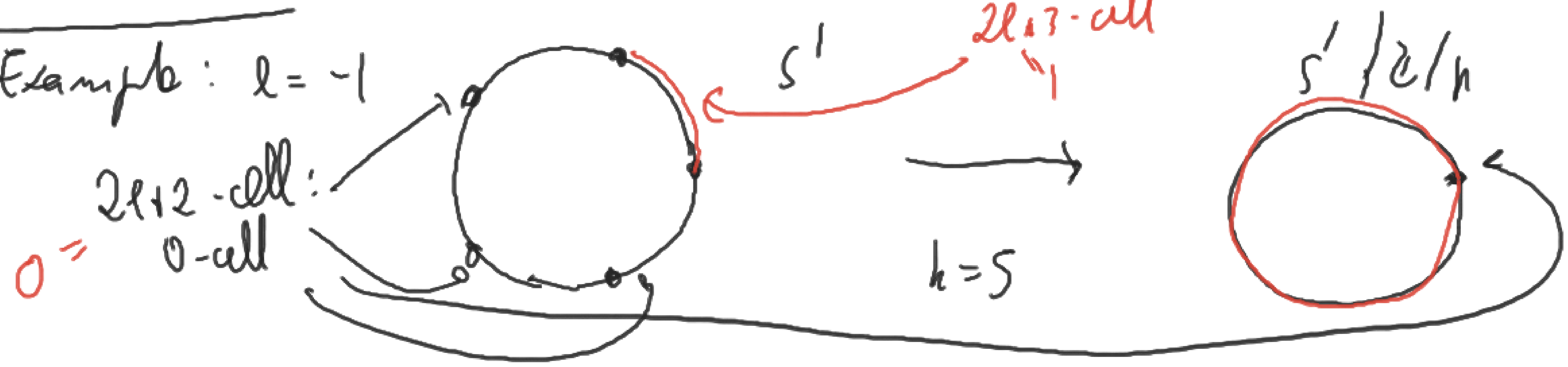
$0 < t < 1$, $s = 0$ or 1 to be in $S^{2l+2} < D^{2l+3}$

$(t z_0, \dots t z_l, (1-t), 0, \dots, 0, s)$ to $(t z_0, \dots t z_l, 1-t) \in (2l+2)$-dim. cell.

Note: In $L_{2\ell+3}(k)$, in the case $0 \le t \le 1$,

$$(tz_0, \dots tz_\ell, (1-t), 0, \dots 0, s) \longmapsto (tz_0, \dots tz_\ell, (1-t)e^{\frac{2\pi i}{k}s}, 0, \dots 0)$$

Example: $\ell = -1$



Calculating homology of $L_{2n+1}(k)$:

$$\underset{2n+1}{\mathbb{Z}} \xrightarrow{0} \underset{2n}{\mathbb{Z}} \xrightarrow{k} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{k} \mathbb{Z} \xrightarrow{0} \underset{0}{\mathbb{Z}}$$

$d_{2m+1} = 0$, $0 \le m \le n$

$d_{2m} = k$ $0 < m \le n$

(since the projection ⊛ has degree k, as the action of $\mathbb{Z}/k$ preserves orientation)

$$\therefore \quad H_m L_{2n+1}(k) = \begin{cases} \mathbb{Z} & m = 0 \quad 2n+1 \\ \mathbb{Z}/k & 0 < m < 2n+1 \text{ odd} \\ 0 & \text{else} \end{cases}$$

$\operatorname{colim} L_{2n+1}(k) = B\mathbb{Z}/k$.

$$\therefore \quad H_m B\mathbb{Z}/k = H_m \mathbb{Z}/k = \begin{cases} \mathbb{Z} & m = 0 \\ \mathbb{Z}/k & m > 0 \text{ odd} \\ 0 & \text{else.} \end{cases}$$

<u>Example</u> (Quillen): For a group G, we write

$G' =$ subgroup generated by $aba^{-1}b^{-1}$, $a,b \in G$
(normal)

$G/G' = G^{ab}$

Suppose X is a connected CW-complex such that $G = \pi_1 X$,

$G'' = G'$.

(example: Σ_n = symm. group on n elements, $n \geqslant 5$.)

$\Sigma_n' = A_n \qquad A_n' = A_n$.

Theorem: There exists a CW-complex Z, $X \overset{i}{\subseteq} Z$ where $\pi_1 Z = (\pi_1 X)^{Ab}$, i induces an isomorphism in homology.
($Z = X^+$, plus-construction)

Proof: First form a space Y by attaching 2-cells e_i to X along generators of $\pi_1(X)'$. In fact, note that in the long exact sequence of the pair (Y, X), $\partial e_i = 0$ (because we have $H_1(X) \to H_1(Y)$. But we can say more:

$$\begin{array}{c} H_1(X) \\ \| \\ \pi_1(X)/\pi_1'(X) \end{array} \quad =$$

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d, "\pi_X"] \arrow[dr, phantom, "\lrcorner", very near end] & \tilde{Y} \arrow[d, "\pi"] \\
X \arrow[r, "\subseteq"', "i"'] & Y
\end{tikzcd}$$

universal covering

the covering restricting π to X.

$$\pi_1(\tilde{X}) = \pi_1(X)'.$$

HW 5 Prove that if we have a pullback of based coverings π, π_X

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\lrcorner", very near end] & \tilde{Y} \arrow[d, "\pi"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

(based connected CW-complexes) $\pi_1(Y, *) = G$, $\pi_1(\tilde{Y}, *) = H \subseteq G$, then considering $f_* : \pi_1(X, *) \to \pi_1(Y, *)$, we have $\pi_1(\tilde{X}, *) = f_*^{-1}(H)$.

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "\pi_X"'] & \tilde{Y} \arrow[d, "\pi"] \\ X \arrow[r] & Y \end{tikzcd}$$

Fundamental groups: $\pi_1(X, *) = G$

$$\begin{tikzcd} G' \arrow[r] \arrow[d] & 0 \arrow[d] \\ G \arrow[r] & G^{ab} \end{tikzcd}$$

The attaching maps of the cells e_i lift to $\tilde{X}$, $H_1(\tilde{X}) = 0$

$\therefore$ The lifts of the cells e_i represent elements of $H_2 \tilde{Y}$.

By LES of $(\tilde{Y}, \tilde{X})$

But $\pi_1(\tilde{Y}) = 0$. Therefore, those lifts represent element of $\pi_2(\tilde{Y})$. $\therefore$ The 2cells e_i represent element of $\pi_2(Y)$ (by projection).

$X^+ = Z$ is formed by attaching 3-cells to those element of $\pi_2(Y)$. $\square$