MATH 592 4/1/2024

X any path-connected space.

$$h : \pi_n(X) \longrightarrow H_n(X) \qquad \text{Hurewicz map}$$

$$\rotatebox{90}{$\in$}$$

$$[\alpha : S^n \longrightarrow X]$$

$$H_n\alpha : \underbrace{H_n S^n}_{\cong\,\mathbb{Z}} \to H_n X$$

$$1 \longmapsto h[\alpha]$$

<u>Hurewicz Theorem</u>: Suppose X is path-connected, $\pi_i(X) = 0$ for $i < k$. Then $h : \pi_k X \to H_k X$ is an isomorphism for $k \geq 2$. (abelianization for $k = 1$).

Special case relevant to + construction:

<u>Lemma</u>: If X is a simply connected CW-complex then
$$h : \pi_2 X \to H_2 X$$
is onto.

<u>Proof</u>: Let $\tilde{X}_1$ be obtained from X_1 by attaching disks to free generators of $\pi_1 X_1$ (free group). Then $\tilde{X}_1 \simeq *$.

$$\begin{tikzcd} X_1 \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap" rotate=90] & X \\ \tilde{X}_1 \arrow[ur, dotted] & \end{tikzcd}$$

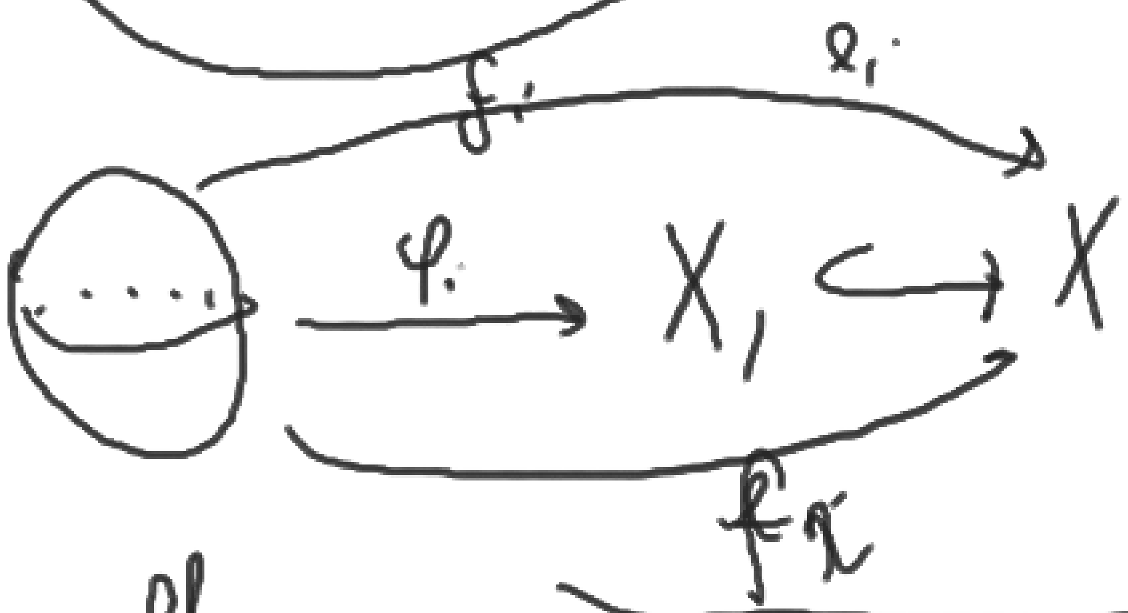
$\longleftarrow$ extends because $\pi_1 X = 0$.

For a 2-cell $e_i : D^2 \to X$ $\quad$ ($e_i|_{S^1} \xrightarrow{\varphi_i} X_1$ attaching map)

$$\begin{tikzcd}
S^1 \arrow[r, "\varphi_i"] \arrow[d, hook] & X_1 \arrow[d] \arrow[dr, bend left] & \\
D^2 \arrow[r, dotted, "\bar{e}_i"] \arrow[rr, bend right, "f_i"'] & \tilde{X}_1 \arrow[r] & X
\end{tikzcd}$$

contractible

Glue e_i with f_i.



Call this map $\psi_i : S^2 \to X$

Now if $\sum n_i e_i \in C_2^{cell} X$,
$d^{cell}(\sum n_i e_i) = 0$

$$\boxed{h(\sum n_i \psi_i) = [\sum n_i e_i]}$$

(from the point of view of sing. homology, $d \sum n_i f_i = 0$ and lifts to $\tilde{X}_1 \simeq *$.)

The + construction. (Quillen ~ 1970)

Let X be a connected CW-complex, assume $(\pi_1 X)'' = (\pi_1 X)'$.

choose a lift $\tilde{e}_i$ of each 2-cell e_i

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] \\
X \arrow[r, hook] & Y
\end{tikzcd}$$

cover $\rightarrow$; $\leftarrow$ universal cover

$\pi_1(\tilde{X}, *) = (\pi_1 X)'$

attach 2cells e_i to generators of $(\pi_1 X)'$

$$H_1 \tilde{X} = (\pi_1 X)'/(\pi_1 X)'' = 0.$$

The attaching map of $\tilde{e}_i$ to $\tilde{X}$ induces 0 in homology

$$\begin{tikzcd}
\bar{e}_i \arrow[r, mapsto] & \tilde{e}_i \arrow[rr, mapsto] & & 0 \\
H_2(\tilde{Y}) \arrow[r] & H_2(\tilde{Y}, \tilde{X}) \arrow[r, "\partial"] & H_1 \tilde{X} &
\end{tikzcd}$$

with $\bar{e}_i \in H_2(\tilde{Y})$, $\tilde{e}_i \in H_2(\tilde{Y}, \tilde{X})$.

By the lemma, $\bar{e}_i = h(\psi_i)$ $\quad \psi_i : S^2 \to \tilde{Y}$.

Recipe: Attach a 3-cell to each ψ_i. using it as attaching map.

The resulting space is denoted by X^+.

<u>To recap</u>:

$$\begin{array}{ccccc} \bar{\bar{e}}_i & \longmapsto & e_i & \longmapsto & 0 \\ \cap & & \cap & & \\ H_2(Y) & \longrightarrow & H_2(Y,X) & \longrightarrow & H_1(X) \end{array}$$

Y is obtained by attaching 2-cells to X (attaching maps $S^1 \to X_1$).

There exists $\bar{\bar{\psi}}_i \in \pi_2(Y)$ where $h(\bar{\bar{\psi}}_i) = \bar{\bar{e}}_i$. X^+ is obtained by attaching a 3-cell along each attaching map $\bar{\psi}_i$.

HW (1) Prove that the inclusion $X \hookrightarrow X^+$ induces an isomorphism in homology.

Remark: Quillen's use of the $+$-construction:

Suppose R is a "nice ring" (e.g. a field). Consider the discrete group $Gl_n R$. Form $\operatorname{colim} Gl_n R =: Gl_\infty R$.

$$Gl_n R \hookrightarrow Gl_{n+1} R$$
$$A \longmapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}.$$

If R is "nice" (e.g. a field) then $(Gl_n R)'' = (Gl_n R)'$.

Quillen: For $n \geq 1$

$$K_n R := \pi_n\, B(Gl_\infty R)^+ \qquad \boxed{\begin{array}{l} \pi_1 B Gl_\infty R = Gl_\infty R \\ \pi_k B Gl_\infty R = 0 \quad k > 1. \end{array}}$$

(higher) algebraic K-theory.

Mayer-Vietoris sequence: Suppose $U, V \subseteq X$ open, $U \cup V = X$.
$\mathcal{U} = \{U, V\}$. We have a short exact sequence of singular chain complexes

$$0 \to C(U \cap V) \xrightarrow{\begin{pmatrix} \subseteq \\ \subseteq \end{pmatrix}} C(U) \oplus C(V) \xrightarrow{(\subseteq \ -\subseteq)} C_{\mathcal{U}} X \to 0$$

<u>Theorem</u>: We have a long exact sequence

$$\to H_n(U \cap V) \xrightarrow{\begin{pmatrix} \subseteq_* \\ \subseteq_* \end{pmatrix}} H_n(U) \oplus H_n(V) \xrightarrow{(\subseteq_* \ -\subseteq_*)} H_n X \xrightarrow{\partial} H_{n-1}(U \cap V) \to \cdots$$

(∂ natural) $\quad X \xrightarrow{f} X'$
$U \mapsto U'$
$V \mapsto V'$ open $U' \cup V' = X'$

$$\begin{array}{ccc} H_n X & \xrightarrow{\partial} & H_{n-1}(U \cap V) \\ f_* \downarrow & & \downarrow f_* \\ H_n X' & \xrightarrow{\partial} & H_{n-1}(U' \cap V') \end{array} \quad \square$$

<u>Example</u>: Calculate the homology of the complement of the trefoil knot  $\subset S^3$.

$T = S^1 \times S^1 \subset S^3 = S^1 \times D^2 \cup D^2 \times S^1$

$\simeq$

U, V are "thickenings" of $(S^1 \times D^2) \setminus \mathrm{Im}\,\varphi$, $(D^2 \times S^1) \setminus \mathrm{Im}\,\varphi$

$U \cap V \simeq S^1 \times S^1 \setminus \mathrm{Im}\,\varphi = S^1$.

Recall for π_1:

$$\begin{tikzcd} \pi_1(U\cap V) \arrow[r] \arrow[d] & \pi_1 U \\ \pi_1 V & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[r, "2"] \arrow[d, "3"] & \mathbb{Z} \\ \mathbb{Z} & \end{tikzcd}$$

$$\pi_1(\underbrace{U\cup V}_{Y}) = \langle a, b \mid a^2 b^{-3} \rangle$$

$$\downarrow \qquad \therefore \text{ knot non-trivial}$$

$$\Sigma_3$$

$$\begin{tikzcd} H_1(U\cap V) \arrow[r] \arrow[d] & H_1(U) \\ \pi_1(V) & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[r, "2"] \arrow[d, "3"] & \mathbb{Z} \\ \mathbb{Z} & \end{tikzcd}$$

$$\to H_{k+1} Y \xrightarrow{\partial} H_k(U\cap V) \to H_n(U) \oplus H_n(V) \to H_k(Y) \xrightarrow{0} H_{n-1}(U\cap V)$$

We can also work in reduced homology: By Noether Theorem, factor out $x \in U\cap V$

$$0 \to C_* \to C_* \oplus C_* \to C_* \to 0.$$

LES also in reduced homology:

$$\xrightarrow{0} \tilde{H}_k(U\cap V) \to \tilde{H}_k(U)\oplus\tilde{H}_k(V) \to \tilde{H}_k(U\cup V) \xrightarrow{0} \tilde{H}_{k-1}(U\cap V) \to \dots$$

0 for $k \neq 1$

$$\tilde{H}_1(U\cap V) \to \tilde{H}_1(U)\oplus\tilde{H}_1(V)$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix}2\\3\end{pmatrix}} \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z} \to 0$$

Ker $= 0$ Coker $= \mathbb{Z}$

$$\begin{pmatrix}2\\3\end{pmatrix} \sim \begin{pmatrix}1\\0\end{pmatrix}$$

<u>Conclusion</u>: $\tilde{H}_1 Y = \mathbb{Z}$ $\tilde{H}_k Y = 0$ $k \neq 1$ | $H_k Y = \mathbb{Z}$ $k = 0, 1$; 0 else.

HW ② Calculate the homology of the pushout

$$\begin{tikzcd} \mathbb{RP}^k \arrow[r, hook] \arrow[d, hook] & \mathbb{RP}^n \\ \mathbb{RP}^m & \end{tikzcd}$$

discussion tomorrow:
More examples of calculating homology using Mayer-Vietoris sequence

$k \le m$. [Use the Mayer-Vietoris sequence.]

③ ⓐ Prove that a short exact sequence of the form

free abelian group on S

$$0 \to A \to B \to \mathbb{Z}S \to 0$$

splits. ($\therefore B \cong A \oplus \mathbb{Z}S$)

ⓑ Give an example of a short exact sequence of ab. groups which does not split.