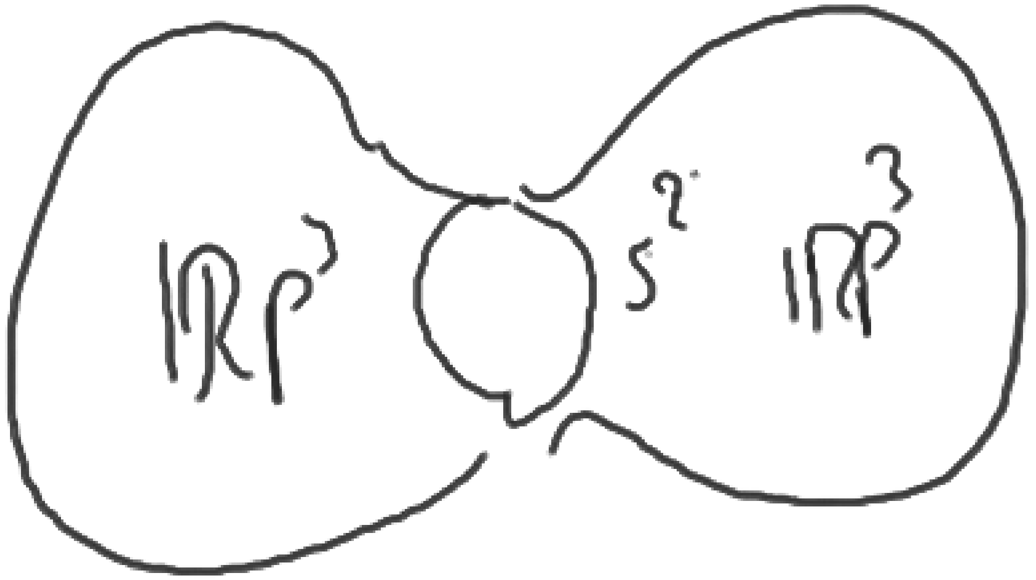
MATH 592 4/3/2024

Example: $(\mathbb{R}P^3 \setminus \text{open disk}) \cup_{\text{boundary}} (\mathbb{R}P^3 \setminus \text{open disk}) = X$

? $H_* X$

"$\mathbb{R}P^3 \# \mathbb{R}P^3$")

for manifolds of dimension ≥ 3, this depends on orientation



$\mathbb{R}P^3 \setminus \text{open disk} \simeq \mathbb{R}P^2$

$S^2 \xrightarrow{\text{covering}} \mathbb{R}P^2$

$$\begin{tikzcd} S^2 \arrow[r] \arrow[d] & \mathbb{RP}^2 \\ \mathbb{RP}^2 & \end{tikzcd}$$

"homotopy pushout"

$$\text{colim}\left(\begin{array}{ccc} S^2 & \longrightarrow & \mathbb{RP}^2 \\ (?,0)\cap & & \\ S^2\times[0,1] & & \\ (?,1)\cup & & \\ S^2 & \longrightarrow & \mathbb{RP}^2 \end{array}\right)$$

Thickening to def. retract,

U, V = thickenings of the copies of $\mathbb{RP}^2 \setminus$ open disk

$$\to \tilde{H}_n(\underbrace{U\cap V}_{\simeq S^2}) \to \tilde{H}_n(\underbrace{U}_{\mathbb{RP}^2}) \oplus \tilde{H}_n(\underbrace{V}_{\mathbb{RP}^2}) \to \tilde{H}_n(U\cup V) \xrightarrow{0} \tilde{H}_{n-1}(U\cap V) \to$$

$n=2$: $\mathbb{Z} \to \mathbb{Z} \to 0$, $\quad \mathbb{Z} = \tilde{H}_3(U\cup V)$

$n=1$: $0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2\oplus\mathbb{Z}/2 = \tilde{H}_1(U\cup V)$

Answer: $H_0 X = \mathbb{Z}$

$H_1 X = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

$H_3 X = \mathbb{Z}$

$H_i X = 0 \quad i \neq 0,1,3$

In general:

$$H_n(U \cap V) \xrightarrow{\varphi_n} H_n U \oplus H_n V$$

$$\varphi_n = \begin{pmatrix} \subseteq_* \\ \subseteq_* \end{pmatrix}$$

$$0 \to \mathrm{Coker}\, \varphi_{n+1} \to H_n(U \cup V) \to \mathrm{Ker}\, \varphi_n \to 0$$

if this is free abelian, it splits

"finite CW-cx."

Euler characteristic – defined for a CW-complex X with finitely many cells

$$C = C^{cell}X \qquad \cdots 0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to 0 \cdots$$

complex of free abelian groups, all finitely generated

$$H_* X = H_n C \quad H_{n-1} C \quad H_0 C$$

Definition: $\chi(X) = \sum_{k \in \mathbb{Z}} (-1)^k \operatorname{rank} \underbrace{H_k X}_{H_k C}$

Euler characteristic

$$\boxed{\operatorname{rank}\left(\mathbb{Z}^n \oplus \mathbb{Z}/_{m_1} \oplus \cdots \oplus \mathbb{Z}/_{m_k}\right) = n \qquad m_1 | \cdots | m_k \neq 0}$$

If $X \simeq Y$ then $\chi(X) = \chi(Y)$

$\oplus C_n$ is f.g.

Proposition: Let C be a finitely generated chain complex of free abelian groups. Then

$$\sum_{k\in\mathbb{Z}} (-1)^k \operatorname{rank} H_k C = \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{rank} C_k .$$

Proof: $\operatorname{rank} H_k C = \operatorname{rank} C_k - \underbrace{\operatorname{rank} d_k - \operatorname{rank} d_{k+1}}_{\text{These cancel out in alternating sum.}}$. $\square$

Examples: $\chi(S^n) = 1 + (-1)^n = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

$\chi(\mathbb{RP}^n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

<u>Proposition</u>: If $Y \longrightarrow X$ is a covering of degree k then

$$\chi(Y) = k\,\chi(X). \qquad \square$$

(k cells over each cell.)

$$\chi(\underbrace{T \# \cdots \# T}_{n}) = 2-2n$$

"genus"

$$\chi(\underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n}) = 2-n$$

"genus" $= n-1$ (?)

H_2	$\mathbb{Z}$
H_1	$\bigoplus_{2n} \mathbb{Z}$
H_0	$\mathbb{Z}$
H_2	0
H_1	$\bigoplus_{n-1} \mathbb{Z} \oplus \mathbb{Z}/2$
H_0	$\mathbb{Z}$

Example: Classify a 3-fold covering Y of $T \# T$.

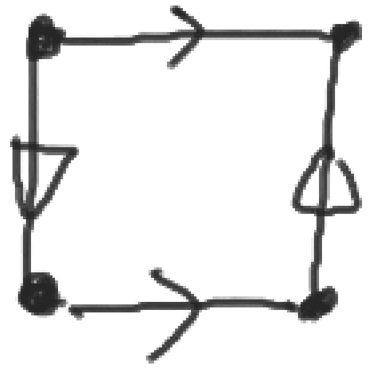
$\chi(T\#T) = -2$

$\chi(Y) = -6$ $\qquad Y \cong T\#T\#T\#T$

HW 4 Classify a k-fold cover of $\#_n T$

Example: Classify coverings of degree 2 of $\mathbb{R}P^2 \# \mathbb{R}P^2$

$\mathbb{RP}^2 \# \mathbb{RP}^2$ = Klein bottle



Homology

$\exists$ 2-fold cover

$T \to \mathbb{RP}^2 \# \mathbb{RP}^2$

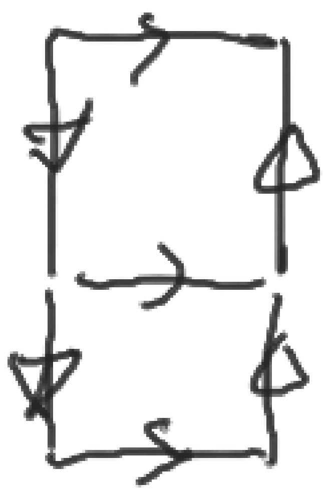
$$\left\{ \begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\begin{pmatrix} 0 \\ 2 \end{pmatrix}} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\ & & & & \\ 0 & & \mathbb{Z}\oplus\mathbb{Z}/2 & & \mathbb{Z} \end{array} \right.$$

Any manifold M has a 2-fold cover $\tilde{M}$ called the orientation cover: $\tilde{M} = \{(x, \text{orientation at } x)\}$ of TM_x if smooth

If M is connected non-orientable then the orientation cover $\tilde{M}$ is oriented, connected.

T is the orientation cover of $\mathbb{RP}^2 \# \mathbb{RP}^2$.

But we can also make a 2-fold cover which remains non-orientable:



$$\mathbb{RP}^2 \# \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2 \# \mathbb{RP}^2.$$

<u>Answer</u>: 2-fold covers of $\mathbb{RP}^2 \# \mathbb{RP}^2$:

T, $\mathbb{RP}^2 \# \mathbb{RP}^2$.

$$\chi(\mathbb{RP}^2 \# \mathbb{RP}^2) = 0 = \chi(T)$$

HW 5 Classify k-fold covers of $\#_n \mathbb{R}P^2$.

<u>Example</u>: ⓐ Every continuous map $f: D^n \to D^n$ has a fixed point.

ⓑ There does not exist a retraction $D^n \underset{\underset{i}{\supset}}{\overset{r}{\rightleftarrows}} S^{n-1}$!
(left inverse $ri = \mathrm{Id}$)

[ⓑ: Any functor preserves retractions.)
H_{n-1}

$\mathbb{Z}$ is not a retract of 0.

$\mathbb{Z} \oplus \mathbb{Z}$ is not a retract of $\mathbb{Z}$ for $n = 1$]

$f : D^n \to D^n$

$f(x) \neq x \quad \forall x \in D^n$

where the open ray from $f(x)$ to x hits S^{n-1}

is a retraction $D^n \to S^{n-1}$!

<u>Lefschetz fixed point theorem</u>: Suppose $f : X \to X$ is a continuous map where X is a finite CW-complex. Suppose

$$0 \neq \lambda(f) = \sum_{n \in \mathbb{Z}} (-1)^n \operatorname{tr} H_n f$$

Then f has a fixed point.