MATH 592 4/5/2024

Degree of a map

Theorem (Hopf) Let M be a compact connected oriented smooth n-manifold, $n \geq 1$. Then the set $[M, S^n] = \mathrm{Mor}_{hTop}(M, S^n)$ homotopy classes of continuous maps $M \to S^n$ is bijective to $\mathbb{Z}$ via a map

$$\deg : [M, S^n] \to \mathbb{Z}$$

defined as follows: let $f : M \to S^n$. Then $f \simeq g$ which has a regular value $x \in S^n$: $\exists\, U \ni x$ open where $f^{-1}(U) = U_1 \amalg \cdots \amalg U_k$, $f|U_i : U_i \xrightarrow{\cong} U$ is a diffeomorphism $y_i \mapsto x$

$$\deg(f) = \sum_{i=1}^{k} \operatorname{sign} Df_{y_i}$$

Explanation: Df_{y_i} :
$$\begin{tikzcd} TM_{y_i} \arrow[r, "\cong"] \arrow[d, phantom, "\wr\|"] & TS^n_x \arrow[d, phantom, "\wr\|"] \\ \mathbb{R}^n \arrow[r, "\cong", "\varphi_i"'] & \mathbb{R}^n \end{tikzcd}$$

choose oriented isomorphisms

$\operatorname{sign} Df_{y_i} := \operatorname{sign} \det(\varphi_i)$.

<u>Note</u>: We only need $M = S^n$.

<u>Corollary</u>: $\pi_n(S^n) = \mathbb{Z}$.

WLOG $n > 1$. If $g, f : S^n \to S^n$ are based h.f. $\simeq g$ unbased then also $f \simeq g$ based.

$n>1$

$\Pi_n : \Pi X \to Ab$ $\qquad$ $\pi_1(S^n) = 0$. $\qquad$ $\omega : [0,1] \to S^n$

$\therefore \pi_1(X,*)$ act on $\pi_n(X,*)$ $\qquad$ $\omega(t) = h_t(*)$.

Are we adding correctly in $\mathbb{Z}$? (Postpone.)

Milnor: Topology of a differential viewpoint

To start proving Hopf's theorem:

Step ① : consistency : The definition does not depend on the choice of g, x.

Local: $g|U_1 \sqcup \cdots \sqcup U_k \longrightarrow U$ $\qquad g|U_i \xrightarrow{\text{diffeo}} U$

$x \in U$. The definition of deg does not depend on the choice of $x \in U$.

Step ② : homotopy.

M  $\xrightarrow{g,g'}$  x regular value for both g, g'

$g \simeq g'$

$h: M \times [0,1] \longrightarrow S^n$ $\quad h(x,0) = g(x) \quad h(x,1) = g'(x)$

h not smooth in any way.

<u>Lemma</u>: There exists a homotopy h' which satisfies $h'(x,0) = g(x)$
$h'(x,1) = g'(x)$

there exists an open neighborhood $V \subseteq U$ such that

$$h'|_{h'^{-1}(V)} \to V \text{ is smooth.} \qquad \square$$

(partition of unity)

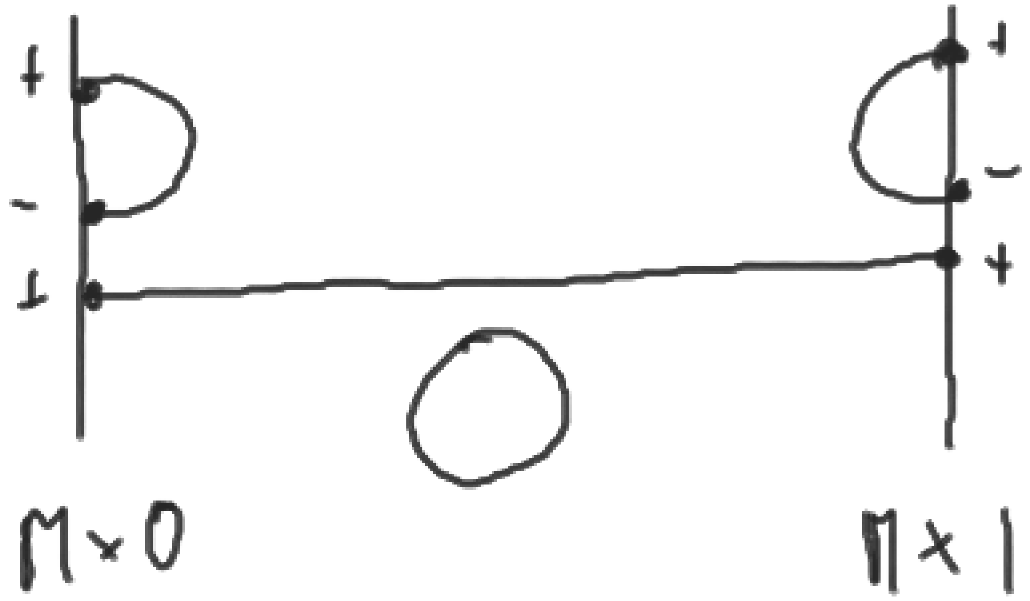
Note: We may not be able to conclude that $h' \pitchfork x$ (D onto org$(h')^{-1}(x)$ $(TM \times [0,1])_y$)

But by Sard's thm, $\exists y \in V$ $h' \pitchfork y$. WOLOG, $x = y$. (By step 1).

Then $(h')^{-1}(y)$ is a smooth compact 1-manifold with boundary.

Classification of smooth compact 1-manifolds with boundary: $\coprod S^1$, $[0,1]$.

Riemann metric
parametrisation by arc length



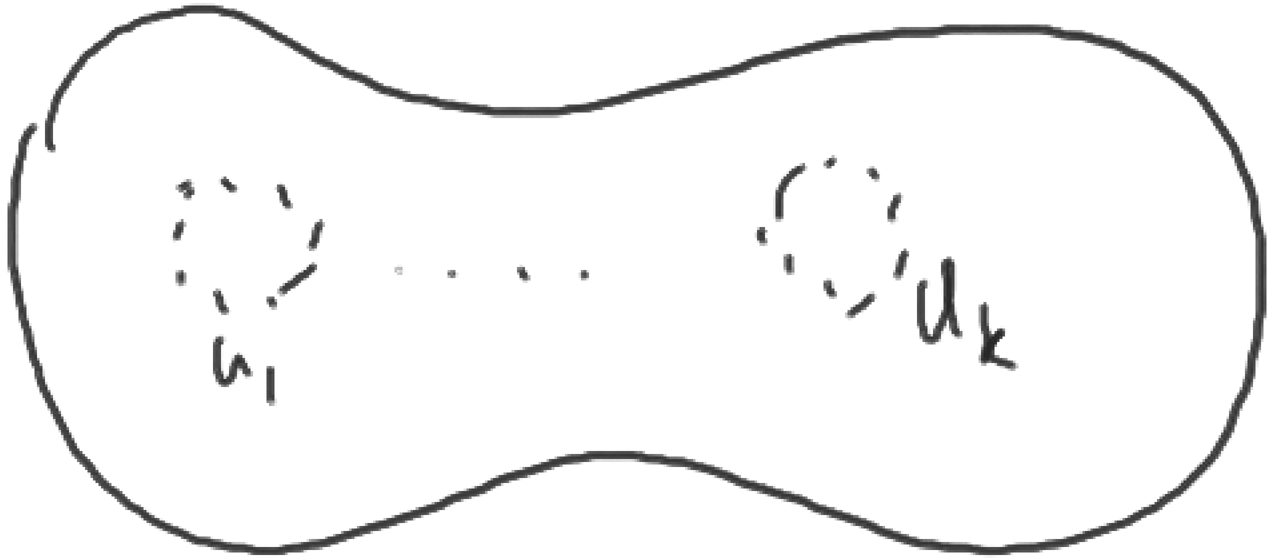
$(h')^{-1}(y)$

Possible scenarios: degrees at g, g', y are the same.

Step ③ : deg is injective ($\deg f = \deg f'$ $f, f' : M \to S^n$
$\Rightarrow f \simeq f'$).

Take f
WOLOG, we have a regular value $U \ni x$ ← open disk

$$f^{-1}(U) = U_1 \sqcup \cdots\cdots \sqcup U_k \qquad f|_{U_i} : U_i \xrightarrow{\text{diffeo}} U$$



$* \notin U$

D^n
$\cong$
$S^n \setminus U \simeq *$

Possibly by replacing U with $V \subseteq U$
↖ $\cong$ open disk

$f \simeq f''$ $f'|_{M \setminus (U_1 \sqcup \cdots \sqcup U_n)} = \text{const}_*$

We can choose coordinates in such a way that $f'|_{U_i}$ is linear.

Lemma: $\pi_0(GL_n\mathbb{R}) = \mathbb{Z}/2$
$= \{+,-\}$

Proof:

Gauss elimination:

- add a multiple of one row to another $\simeq \mathrm{Id}$ via $\in GL_n$
- multiply a row by $\lambda > 0$ $\simeq \mathrm{Id}$ via $\in GL_n$

D^n/S^{n-1}
$= (\mathbb{R}^n)^0$

$\mathbb{R}^n \xrightarrow{\text{linear}} \mathbb{R}^n$

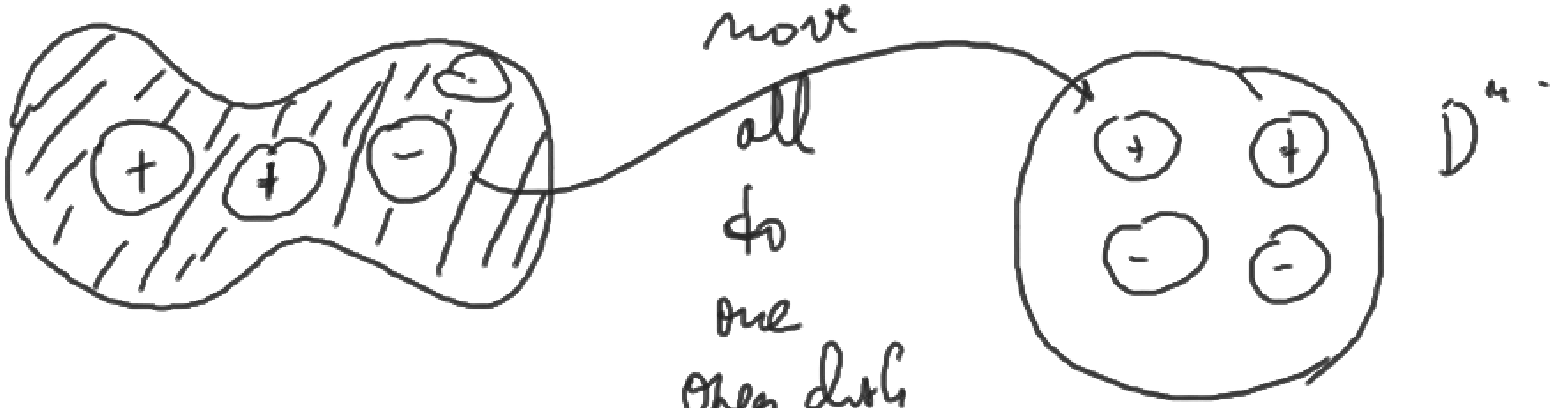
$A \simeq \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & -1 & & \\ & & & -1 & \\ 0 & & & & \ddots \end{pmatrix}$ 1 or -1

$\mathbb{R}^2 = \mathbb{C}$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \simeq_{GL_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\ldots \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$. $\square$

det.



$\leftarrow$ $(n-1)$-suspension of the situation at $n=1$

Step ④ onto

$\square$

Example: A ~~polynomial~~ non-constant polynomial in a complex variable

$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ $\qquad a_k \in \mathbb{C} \qquad \underline{a_n \neq 0.}$

This defines a map $S^2 \longrightarrow S^2$.

$$S^2 = \mathbb{P}^1\mathbb{C} = \mathbb{C}^0 = \mathbb{C} \cup \{\infty\}$$

$\simeq z^n$ (by considering $|z| >> 0$)

$\deg f = n$

$\therefore$ $\underline{f \text{ has a root}}$

HW 6 $f(z) = \dfrac{a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0}{b_n z^n + b_{n-1} z^{n-1} + \cdots b_0} : S^2 \longrightarrow S^2$ $\qquad a_m, b_n \neq 0.$

$$S^2 = \mathbb{C} \cup \{\infty\} \longrightarrow S^2 = \mathbb{C} \cup \{\infty\}$$

What is $\deg f$?