MATH 592 4/8/2024

<u>Lefschetz fixed point theorem</u> (Hatcher: Algebraic Topology)

Let A be a finitely generated abelian group, $f : A \longrightarrow A$ be a homomorphism. We want to define tr f.

For any abelian group A, we have a short exact sequence

$$0 \to A_{tor} \xrightarrow{\subseteq} A \longrightarrow A_{non-tor} \to 0 \qquad \circledast$$

$A_{tor} = \{x \in A \mid \exists n \in \mathbb{N}\} \quad nx = 0.$

$A_{non-tor}$ is <u>torsion-free</u>

G torsion-free: $\forall g \in G \ \forall n \in \mathbb{N} \quad ng = 0 \Rightarrow g = 0.$

If A is finitely generated then $A_{non-tor}$ is free abelian, so the SES $\circledast$ splits.

$$A = A_{tor} \oplus A_{non\text{-}tor}$$

$$\begin{array}{cc|c|c}
 & & \multicolumn{2}{c}{\overbrace{\hspace{6em}}^{\text{source}}} \\
 & f & A_{tor} & A_{non\text{-}tor} = \mathbb{Z}^{\ell} \\
\hline
\text{target} & A_{tor} & ? & ? \\
\hline
 & A_{non\text{-}tor} = \mathbb{Z}^{\ell} & 0 & \boxed{?}
\end{array}$$

$\ell \times \ell$ matrix B of integers

$$\operatorname{tr} f := \operatorname{tr} B \in \mathbb{Z}$$

Rationalization: $(?)_{\mathbb{Q}} : Ab \to \mathbb{Q}\text{-Vect}$ ← category of $\mathbb{Q}$-vector spaces, homomorphism

$$A_{\mathbb{Q}} = \operatorname{colim}\left(A \xrightarrow{2} A \xrightarrow{3} A \xrightarrow{4} A \to \cdots A \xrightarrow{n} A \to \cdots\right)$$

exact functor.

$$A \text{ torsion} \Rightarrow A_{\mathbb{Q}} = 0$$

$$A_{\mathbb{Q}} = (A_{\text{non-torsion}})_{\mathbb{Q}}.$$

We can also define for $f : A \to A$

$$\operatorname{tr} f = \operatorname{tr}(f_{\mathbb{Q}} : A_{\mathbb{Q}} \to A_{\mathbb{Q}})$$

(note: By the above discussion, we know $\operatorname{tr} f \in \mathbb{Z}$!)

<u>Lefschetz fixed point theorem</u>: let X be a finite CW-complex, let $f : X \to X$ be a continuous map. Define

$$\lambda(f) = \sum_{n \in \mathbb{Z}} (-1)^n \operatorname{tr}(H_n f : H_n X \to H_n X)$$

Lefschetz number

If $\lambda(f) \neq 0$ then f has a fixed point. ($\exists x \in X \quad f(x) = x$).

For the proof,

Step 1 (purely algebraic): Let C be a finitely generated chain complex (of free abelian groups). Let $f: C \to C$ be a chain map.

Then
$$\sum_{n\in\mathbb{Z}} (-1)^n \operatorname{tr}(H_n f : H_n C \to H_n C) = \sum_{n\in\mathbb{Z}} (-1)^n \operatorname{tr}(f_n : C_n \to C_n).$$

This follows from

<u>Lemma</u>:
$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & D & \to & 0 \\ & & \downarrow g_A & & \downarrow g_B & & \downarrow g_D & & \\ 0 & \to & A & \to & B & \to & D & \to & 0 \end{array}$$

A, B, D f.g. abelian groups.

$\operatorname{tr} g_B = \operatorname{tr} g_A + \operatorname{tr} g_D$.

HW 1: Prove this lemma.

$$0 \to B_n \to Z_n \to H_n C \to 0$$

$$0 \to \underset{\text{cycles}}{Z_n} \to C_n \to \underset{\text{boundaries}}{B_{n-1}} \to 0$$

Step 2 : If X, Y are CW-complexes, a cellular map $f: X \to Y$ is a continuous map such that $f(X_n) \subseteq Y_n$.
(Such a map induces a chain map $C^{cell}(X) \to C^{cell}(Y)$.)

Theorem : Every continuous map $f: X \to Y$ between CW-complexes is homotopic to a cellular map. $\square$

A ~~compact~~ finite CW-complex is metrizable $(\subseteq \mathbb{R}^N)$

A <u>subdivision</u> of a CW-complex X is a CW-structure X' on the same space such that $Id : X \to X'$ is cellular.

<u>Theorem</u>: Let X be a finite CW-complex and $f : X \to X$ be a continuous map. Choose a metric d on X. Let $\varepsilon > 0$. Then there exists a subdivision X' of X and a cellular map $g : X' \to X'$ such that $\forall x \in X$ $d(f(x), g(x)) < \varepsilon$.

[CW-complexes are locally contractible.], $f \simeq g$ □

<u>Step</u> 3. Let $f : X \to X$, X finite CW-complex. Choose a metric and choose an $\varepsilon > 0$. Find a subdivision X', $g : X' \to X'$ cellular

$\forall x \in X$ $d(f(x), g(x)) < \varepsilon$.

$C_g : CX' \to CX'$ $\operatorname{tr} C_g = \operatorname{tr} H_g = \lambda(f) \neq 0$

$\operatorname{tr} H_g = \operatorname{tr} H_f = \lambda(f)$, since $f \simeq g$

$\exists$ an open cell e of X' such that $g(e) \cap e \neq \emptyset$.
(otherwise, $\operatorname{tr} C_g = 0$.)

Step ④ $\therefore \exists x \in X \;\; d(f(x), x) < 3\varepsilon$.
Standard point set topology $\Rightarrow$ exists a fixed point. $\square$

Discussion tomorrow: Examples

The discussion afterwards: Th Apr 18

<u>Example</u>:

If M is a compact connected smooth manifold and there exists a nowhere vanishing vector field on M, then $\chi(M) = 0$.

Non-vanishing vector field v : integrate it f_ε shift along v $\varepsilon > 0$ small
(metric)

$f_\varepsilon \simeq Id$, f_ε has no fixed point.

$$\therefore \quad \underbrace{\lambda(f_\varepsilon)}_{=0} = \lambda(Id) =_{def} \chi(M). \qquad \square$$

More advanced questions:

- ? For what n is S^n parallelisable ? ($\exists\, n$ everywhere independent vector fields)
- How many independent vector fields are there on S^n ?

(Require <u>K-theory</u> - Math 695)

Example: For what n does $\mathbb{Z}/2$ act freely on $\mathbb{CP}^n$?

Solution:

$H_*(\mathbb{CP}^n)$

dim. 0

$\gamma \in \mathbb{Z}/2$ generator
induces Id on H_0.

$$\underbrace{\mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad \cdots \quad \mathbb{Z}}_{(n+1) \text{ copies of } \mathbb{Z}}$$

$\chi(\mathbb{CP}^n) = n+1$

$\gamma^2 = \mathrm{Id}$ $\qquad H_{2k}\gamma = \pm 1 : \mathbb{Z} \longrightarrow \mathbb{Z}$

n even: λ_γ odd $\neq 0$. Cannot exist by Lefschetz fixed point thm.

For n odd it exists:

$$[z_0 : \dots : z_n] \longmapsto [z_1 : -z_0 : z_3 : -z_2 \ \cdots \ z_n : -z_{n-1}]$$

even number of proj. coords.

HW ② Prove that for $k \in \mathbb{N}$ odd, there never exists a free $\mathbb{Z}/k$-action on $\mathbb{C}P^n$, $n \geq 0$.

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③ Given a $k \in \mathbb{N}$, for what n does there exist a free $\mathbb{Z}/k$-action on $\mathbb{R}P^n$?

[Consider S^n.]