MATH 592 4/10/2024

This week's HW will be the last HW set.

Local homology

A *topological m-manifold* (without boundary) is a topological space X where $\forall x \in X$ $\exists x \in U$ open : $U \cong \mathbb{R}^m$.

If X is a topological space, then its *local homology* at $x \in X$ is

$$H_*(X, X \setminus \{x\}).$$

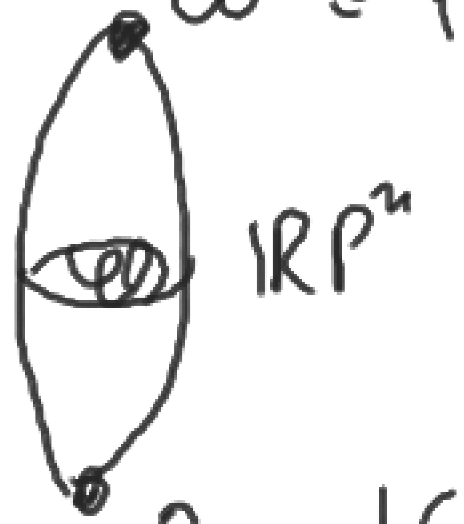
If X is a topological n-manifold, then by excision,

$$H_k(X, X\setminus\{x\}) \cong H_k(U, U\setminus\{x\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) = \begin{cases}\mathbb{Z} & k=n\\ 0 & k\neq n.\end{cases}$$

<u>Example</u>: For which $n \in \mathbb{N}_0$ is the unreduced suspension $S\mathbb{R}P^n$ a topological manifold (without boundary)?

<u>Solution</u>:

$\infty = \{(x,1)\}$



$0 = \{(x,0)\}$

local homology $H_*(S\mathbb{R}P^n, S\mathbb{R}P^n\setminus\{\infty\})$

LES in reduced homology:

$$\overbrace{\tilde H_k(S\mathbb{R}P^n\setminus\{\infty\})} \to \tilde H_k(S\mathbb{R}P^n) \to H_k(S\mathbb{R}P^n, S\mathbb{R}P^n\setminus\{\infty\}) \to \overbrace{\tilde H_{k-1}(S\mathbb{R}P^n\setminus\{\infty\})}^{0}$$

$S\mathbb{R}P^n\setminus\{\infty\} \simeq *$

$$\tilde H_k(S\mathbb{RP}^n) \cong H_k(S\mathbb{RP}^n, S\mathbb{RP}^n \setminus \{\infty\})$$

$$\| $$

$$\tilde H_{k-1}(\mathbb{RP}^n)$$

$$H_k(S\mathbb{RP}^n, S\mathbb{RP}^n \setminus \{\infty\}) = \begin{array}{l} \vdots \quad \leftarrow \mathbb{Z} \text{ if } n \text{ odd} \\ \mathbb{Z}/2 \\ 0 \\ \mathbb{Z}/2 \\ 0 \\ 0 \end{array}$$

$n = 0$  not a top. mfld without boundary.

Only candidate remaining: $n = 1$

$\mathbb{RP}^1 = S^1$ $\qquad$ $S\mathbb{RP}^1 = S^2$

For $n = 1$, yes.

HW 4 The cone CX on a space X is:

$$CX = X \times [0,1] \big/ (x,1) \sim (x',1) \quad , \quad x, x' \in X$$

For what values of n is $C\mathbb{CP}^n \setminus (\mathbb{CP}^n \times \{0\})$ a topological manifold (without boundary)?

5 For what values of n is $\mathbb{R}^n / x \sim -x$ a topological manifold (without boundary)?

Jordan separation theorem

(Massey: Singular homology theory)

<u>Lemma</u>: Let $Y \subset S^n$ be a subset homeomorphic to $[0,1]^k$, $0 \leq k \leq n$. Then

$$\tilde{H}_i(S^n \setminus Y) = 0 \quad \text{for all } i.$$

<u>Proof</u>: Induction on k. $k=0$ $S^n \setminus \{*\} \cong \mathbb{R}^n \simeq *$. OK.

Suppose the statement holds with k replaced by $k-1$. Consider

$$\varphi : [0,1]^k \xrightarrow{\cong} Y,$$

$$Y_0 = \varphi\left([0,1]^{k-1} \times [0, \tfrac{1}{2}]\right), \quad Y_1 = \varphi\left([0,1]^{k-1} \times [\tfrac{1}{2}, 1]\right)$$

$Y_0 \cap Y_1 \cong [0,1]^{k-1}$.

$$\underbrace{\tilde{H}_{i+1}(S^n \setminus (Y_0 \cap Y_1))}_{0} \xrightarrow{\Delta} \tilde{H}_i(S^n \setminus Y) \to \tilde{H}_i(S^n \setminus Y_0) \oplus \tilde{H}_i(S^n \setminus Y_1) \to \underbrace{\tilde{H}_i(S^n \setminus (Y_0 \cap Y_1))}_{0}$$

ind. hypothesis

If $0 \neq x \in \tilde{H}_i(S^n \setminus Y)$ then $x \overset{\iota_*}{\longmapsto} \neq 0 \in \tilde{H}_i(S^n \setminus Y_\varepsilon)$ for some $\varepsilon \in \{0,1\}$.

We can repeat this process, there exists a sequence of intervals $J_1, J_2, \ldots$, J_ℓ of length $1/2^\ell$ where

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_\ell \supseteq \cdots \qquad \iota_*(x) \neq 0 \in \tilde{H}_i(S^n \setminus \varphi([0,1]^{k-1} \times J_\ell)).$$

$\therefore$ by the colimit axiom, $\iota_*(x) \neq 0 \in \tilde{H}_i(S^n \setminus \varphi([0,1]^{k-1} \times \underbrace{\bigcap_\ell J_\ell}_{\text{a single point}}))$

This contradicts the induction hypothesis.

Proposition: let $A \subset S^n$, $A \cong S^k$ for $0 \leq k \leq n-1$. Then

$$\tilde{H}_{n-k-1}(S^n \setminus A) \cong \mathbb{Z}$$

$$\tilde{H}_i(S^n \setminus A) \cong 0 \quad i \neq n-k-1.$$

Corollary (Jordan separation theorem): Suppose $A \subset S^n$, $A \cong S^{n-1}$ then $S^n \setminus A$ has exactly two connected components.

Proof: By the Proposition, $\tilde{H}_0(S^n \setminus A) = \mathbb{Z}$ so $H_0(S^n \setminus A) = \mathbb{Z} \oplus \mathbb{Z}$, so the conclusion follows. $\square$

<u>Proof of Proposition</u>: Induction on k. $\quad S^k = \{ x \in \mathbb{R}^{k+1} \mid \|x\| = 1 \}$

$\varphi : S^k \xrightarrow{\cong} A$ $\quad D^k_\pm = \{ (x_0, \dots, x_k) \in S^k \mid x_k \underset{\le}{\ge} 0 \}$.

$D^k_+ \cup D^k_- = S^k \qquad D^k_+ \cap D^k_- \cong S^{k-1}$. Mayer-Vietoris sequence:

$$\tilde{H}_i(S^n \setminus \overset{=0}{\varphi(D^k_+)}) \oplus \tilde{H}_i(S^n \setminus \varphi(D^k_-)) \to \tilde{H}_i(S^n \setminus \varphi(\underbrace{D^+_k \cap D^-_k}_{S^{k-1}})) \xrightarrow{\Delta} \tilde{H}_{i-1}(S^n \setminus \varphi(S^k)) \to \tilde{H}_{i-1}(S^n \setminus \varphi(D^k_+)) = 0$$

$$\tilde{H}_{i-1}(S^n \setminus \varphi(S^k)) \cong \tilde{H}_i(S^n \setminus \varphi(D^+_k \cap D^-_k)).$$

Use induction hypothesis,

Comment: At this stage, we cannot assume a priori that the $Y \subseteq S^m$, $A \subseteq S^m$ are strict. We should not be using $\tilde{H}$ on an empty set.

The next aim: Invariance of domain: Suppose $U \subseteq \mathbb{R}^n$, $\varphi: U \to \mathbb{R}^n$ is continuous, homeomorphic to its image. Then $\varphi(U) \subseteq \mathbb{R}^n$ is open.

HW 6 Find an example of a CW-complex X where the statement of invariance of domain fails with $\mathbb{R}^n$ replaced by X.

Corollary: There does not exist a homeomorphic embedding $\mathbb{R}^{n+1} \underset{\approx}{\subseteq} \mathbb{R}^n$ (+ obvious variants).

$$\mathbb{R}^{n+1} \underset{\approx}{\subseteq} \mathbb{R}^n \underset{\neq}{\subseteq} \mathbb{R}^{n+1}$$

Similarly: $[0,1]^{n+1} \not\hookrightarrow [0,1]^n$. (even though there is a continuous onto map $[0,1]^n \to [0,1]^{n+1}$ for $n \geqslant 1$).