MATH 592 4/12/2024

<u>Comment</u>: Using reduced homology <u>OK</u>. One only needs to remark that an embedding $D^k \subseteq S^n$, $S^k \subseteq S^n$, $k<n$ cannot be onto. Onto $\Rightarrow$ homeomorphism $\Rightarrow$ $\cong$ in homology. contradiction.

<u>Theorem</u>: Let $A \subset S^n$, $A \cong S^{n-1}$, $n \geq 1$. Denote the connected component of $S^n \setminus A$ by C_1, C_2. Then $\partial C_1 = \partial C_2 = A$.

[$\partial Y = \text{Closure}(Y) \setminus \text{Interior}(Y)$].

<u>Note</u>: There exists an $A \subset S^n$, $A \cong S^{n-1}$, $\mu(A) > 0$.

Lebesgue measure

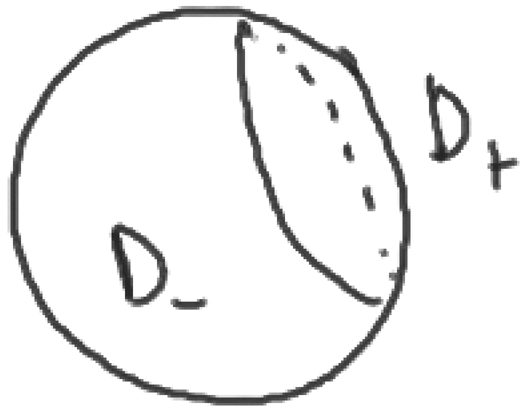
<u>Prove</u>: $\partial C_i \subseteq A$. To prove equality, let $x \in A$. Let N be an open neighborhood of x in S^n. Let

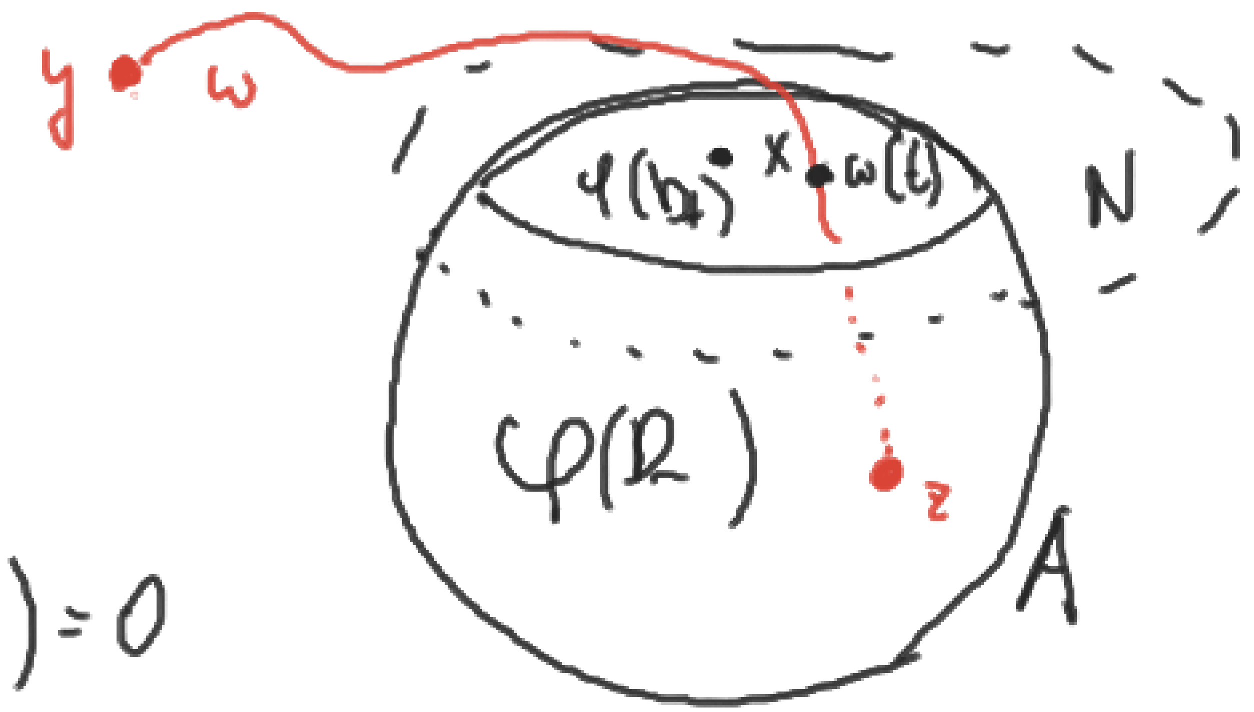
$$\varphi : S^{n-1} \xrightarrow{\cong} A \qquad S^{n-1} = D_+ \cup D_-, \quad D_+ \cap D_- \cong S^{n-2}$$

(say, $n \geq 2$).

Disks, not necessarily the standard ones.

Select D_+, D_-, $\varphi(D_+) \subseteq N$, $x \in \varphi(D_+ \setminus D_-)$





$\tilde{H}_0(S^n \setminus \varphi(D_-)) = 0$

$S^n \setminus \varphi(D_-)$ is path-connected (by the first lemma from last time).

Now let $y \in C_1$, $z \in C_2$. Let $\omega : [0,1] \to S^n \setminus \varphi(D_-)$ $\omega(0) = y$, $\omega(1) = z$

$\exists\, t \in (0,1)$ $\omega(t) \in A$. Let $0 < t_0 \leq t_1 < 1$: $t_0 = \min\{t \mid \varphi(t) \in A)$

$t_1 = \max\{t \mid \varphi(t) \in A\}$

Since N is open, $\omega^{-1}N$ is open in $[0,1]$, $t_0, t_1 \in \omega^{-1}N$. $\exists$ $s_0, s_1 \in \omega^{-1}(N)$

$$0 < s_0 < t_0 \leq t_1 < s_1 < 1$$

$\omega(s_0) \notin A$, $\omega(s_1) \notin A$. $\omega(s_0) \in C_1$, $\omega(s_1) \in C_2$. $\therefore N \cap C_1 \neq \emptyset$, $N \cap C_2 \neq \emptyset$. $\square$

__The invariance of domain__: Let $U \subseteq \mathbb{R}^n$ be open, $\varphi : U \xrightarrow[\subseteq]{} \mathbb{R}^n$ homeomorphic onto its image. ($U \xrightarrow[\cong]{\varphi} \varphi(U)$) ← subspace topology.

Then $\varphi(U) \subseteq \mathbb{R}^n$ is open.

Proof: Let S^n be the 1-point compactification of $\mathbb{R}^n$. Let $x \in U$. Let B be a closed ball (of radius > 0) with center x contained in U.

$$\begin{tikzcd} B \subset U \arrow[rr, "\varphi", "\cong"'] \arrow[rrrr, bend right, "\psi"'] & & V \subseteq \mathbb{R}^n & \subset & S^n \end{tikzcd}$$

$$\partial_U B = \partial_{\mathbb{R}^n} B \cong S^{n-1}.$$

By the Jordan separation theorem, $S^n \setminus \psi(\partial B)$ has exactly two connected components. As sets,

$$S^n \setminus \psi(\partial B) = \left(S^n \setminus \psi(B)\right) \amalg \psi\left(B \setminus \partial B\right) \qquad (*)$$

$\psi : B \longrightarrow S^n$ is injective, $\psi : B \xrightarrow{\cong} \psi(B)$ is a homeomorphism because B is compact. By the first lemma from last time,

$$\tilde{H}_i\left(S^n \setminus \psi(B)\right) = 0 \quad \forall i.$$

Therefore, $S^n \setminus \psi(B)$ is connected. But $\psi(B \setminus \partial B)$ is the image of a connected set, hence connected. So both of the summands of the right hand side of $(*)$ are connected. So in fact, they <u>are</u> the

connected components of $S^n \setminus \psi(\partial B)$. Therefore, $\psi(B \setminus \partial B)$ is open.
(by the previous Thm).

So $\psi(B \setminus \partial B)$ is an open neighborhood of $\psi(x)$ in $V = \psi(U)$. So V is open. $\square$

<u>Schönfliess Theorem</u>: A continuous embedding $\underset{\partial D_2}{\overset{\shortparallel}{S^1}} \hookrightarrow S^2$ extends to a continuous embedding $D^2 \hookrightarrow S^2$.

This fails with 2 replaced by $n > 2$. ("Horned sphere")

Construction of $BG = K(G,1)$

(BG connected CW-complex,

$\pi_1 BG = G$

$\pi_n BG = 0 \quad n > 1$

for all groups G).

This brings up the question: Why did we not talk about simplicial complexes?

(Hilton - Wiley)

High-dimensional geometric topology

PL-mflds & topological manifolds (Kirby-Siebenmann)

Useful in geometry (PL-manifolds)

topological manifolds which are (finite) simplicial complexes.

Milnor: strictly more general than smooth mflds (Milnor-Stasheff: Characteristic classes) Chapter 20

In algebraic topology, we are more interested in semisimplicial and simplicial sets.

Recall the standard simplex $\Delta^n = \{(t_0, \dots, t_n) \mid t_i \geq 0 \quad \sum t_i = 1\}$

Face maps: $\quad \partial_i : \Delta^n \to \Delta^{n+1} \qquad i \in \{0, \dots, n+1\}$

$$(t_0, \dots t_n) \longmapsto (t_0, \dots t_{i-1}, 0, t_i, \dots t_n)$$

$$\Delta^0 \rightrightarrows \Delta^1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \Delta^2 \quad \cdots\cdots$$

What is the category generated by the maps ∂_i?

Isomorphic to the category Δ^+ where

$$\mathrm{Obj}\,\Delta^+ = \{0, 1, 2, \cdots\}$$

$$\mathrm{Mor}_{\Delta^+}(m, n) = \text{ordered injections } \{0, \dots, m\} \to \{0, \dots, n\}.$$

(where the coordinates go)

A <u>semisimplicial set</u> is a functor $S : (\Delta^+)^{op} \to \mathrm{Set}$

opposite category; same object, switch S,T, order of composition reverses

$S_*(n)\ (= S_n) =$ "the set of n-simplices"

Example: The singular set of a topological space X:

$$S_n X = \{\text{singular } n\text{-simplices in } X\} = \{\Delta^n \to X\}.$$

$$\Delta^{n-1} \xrightarrow{\partial_i} \Delta^n \to X$$

$$\partial_i : S_n X \to S_{n-1} X$$

Next time: Use semisimplicial sets to construct BG.

- Uniqueness of BG (up to homotopy equivalence).