MATH 592 4/15/2024

A *semisimplicial set* is a functor $S_\cdot : (\Delta^+)^{Op} \to Set$

$Obj\,\Delta^+ = \mathbb{N}_0$ $\quad Mor_{\Delta^+}(m,n) =$ increasing maps $\{0,\dots m\} \to \{0,\dots n\}$.

Category $(\Delta^+)^{Op}$-Set : Objects = semisimplicial sets
Morphisms = natural transformations

We write $S_n = S_\cdot(n) =$ "the set of n-simplices"

Example : X topological space : $S_\cdot(X) =$ singular set of X

$S_n(X) = \{\Delta^n \longrightarrow X \text{ continuous}\}$

This gives a functor

$$S_\bullet : \mathrm{Top} \longrightarrow (\Delta^+)^{op}\text{-Set}$$

This functor has a left adjoint. "The geometric realization"

$$|S_\bullet| = \coprod_{n\in\mathbb{N}_0} S_n \times \Delta^n \Big/ (\partial_i t, x) \sim (t, \partial_i x)$$

$$\begin{tikzcd} \Delta^{n+1} \\ \Delta^n \arrow[u, "\partial_i"] \end{tikzcd} \qquad \begin{tikzcd} S_{n+1} \arrow[d, "\partial_i"] \\ S_n \end{tikzcd}$$

$\partial_i : n \to n+1$

$(t_0, \dots, t_n) = t \in \Delta^n \qquad x \in S_{n+1}$

$$= \mathrm{coeq}\left(\coprod_{g\in \mathrm{Mor}_{\Delta^+}(m,n)} S_n \times \Delta^m \rightrightarrows \coprod_{n\in \underbrace{\mathrm{Obj}(\Delta^+)}_{\mathbb{N}_0}} S_n \times \Delta^n \right)$$

Comments ① For a semisimplicial set $S_\bullet$, $|S_\bullet|$ is a CW-complex

n-skeleton (only use Δ^k, $k \leq n$)

② For a space X, because $|?|$ is left-adjoint to $S_\bullet$

$$|S_\bullet X| \longrightarrow X$$

((CW-approximation))

Example: (Čech resolution) Let X be a discrete set. $EX_\bullet$: semi-simplicial set

$$EX_n = \underbrace{X \times X \times \cdots \times X}_{n+1} = X^{n+1}$$

$EX_\bullet = X^{\Delta^+}$

Focus: $\partial_i : X^{n+1} \longrightarrow X^n$

projection $(x_0, \ldots x_n) \longmapsto (x_0, \ldots, \hat{x}_i, \ldots x_n)$

omitting the ith coordinate

Lemma: If $X \neq \emptyset$, then $|EX.| \simeq *$. (is contractible)

Proof: Choose a point $* \in X$.

$h_s : |EX.| \longrightarrow |EX.|$

$\coprod X^{n+1} \times \Delta^n / \sim$

$h_0 = Id$, $h_1 = const$

$h_s : (x_0, \dots x_n) \times (t_0, \dots t_n) \longmapsto (*, x_0, \dots x_n) \times (s, (1-s)t_0, \dots (1-s)t_n)$

$h_1 = const_{(*,1)}$. $\square$

Now let G be a discrete group. Then G acts freely on $|EG.| =: EG$.

$$g((g_0, \cdots g_n) \times (t_0, \cdots t_n)) = ((gg_0, \dots, gg_n) \times (t_0, \cdots t_n))$$

Define $BG := EG/G$ (space of orbits)

$EG \longrightarrow BG$ is the universal covering. Therefore

$\pi_1 BG = G$

$\pi_n BG = 0 \quad n \neq 1.$

Proposition: Let BG, $B'G$ be two connected CW-complexes whose π_1 is G and their universal covers are contractible. Then $BG \simeq B'G$.

For a discrete group G, a G-CW-complex is a G-space (space with G-action) defined the same way as a CW-complex, but where G acts on each set I_n of n-cells (attaching maps are <u>equivariant</u> = preserve G-action).

$X = \bigcup X_n \qquad X_{-1} = \emptyset \qquad G\text{-set } I_n$

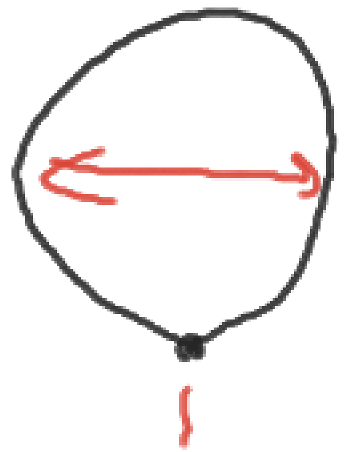
$$I_n \times S^{n-1} \xrightarrow{\varphi_n} X_{n-1}$$

G-equivariant

$$\begin{tikzcd}
I_n \times S^{n-1} \arrow[r] \arrow[d, hook] & X_{n-1} \arrow[d] \\
I_n \times D^n \arrow[r] & X_n
\end{tikzcd}$$

<u>Caution</u>: <u>Not</u> the same thing as G acting on a CW-complex, even by cellular maps

S^1



To prove the proposition, take the universal covers EG, $E'G$ of BG, $B'G$, respectively. Then EG, $E'G$ are free G-CW-complexes.

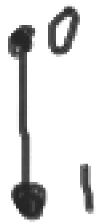
Free means the action of G on each I_n is free (every element has isotropy $\{e\}$).

Forgetting G-action, $EG, E'G \simeq *$.

Construct $EG \xrightarrow[\text{G-equivariant}]{f} E'G$, $E'G \xrightarrow{f'} EG$ by induction on skeleta.

Similarly for homotopies $f \circ f' \simeq Id$, $f' \circ f \simeq Id$

$X \times [0,1]$ is a CW-complex (G-equiv. also)



So we have a G-equivariant homotopy equivalence $EG \simeq E'G$.

Apply $(?)/G$ and we get a homotopy equivalence $BG \simeq B'G$. $\square$

What are <u>simplicial sets</u> and why?

Semisimplicial set too rigid. (?)

<u>Example</u>: $\Delta^n/\partial\Delta^n = S^n$

Further maps between Δ^m, Δ^n:

$s_i : \Delta^n \longrightarrow \Delta^{n-1}$, $i = 1, \dots n-1$ <u>degeneracies</u>

$(t_0, \dots, t_n) \longmapsto (t_0, \dots, t_i + t_{i+1}, \dots t_n)$

What category is generated by faces <u>and</u> degeneracies?

Answer: The <u>simplicial category</u> Δ.

$\mathrm{Obj}\,\Delta = \mathbb{N}_0$

$\mathrm{Mor}_{\Delta}(m, n)$ = Non-decreasing maps $\{0, \dots, m\} \to \{0, \dots, n\}$

(degenerate : not injective)

<u>simplicial set</u> : $\Delta^{op} \to \mathrm{Set}$

category Δ^{op}-Set.

$S_\cdot : \mathrm{Top} \to \Delta^{op}\text{-Set}$ still works

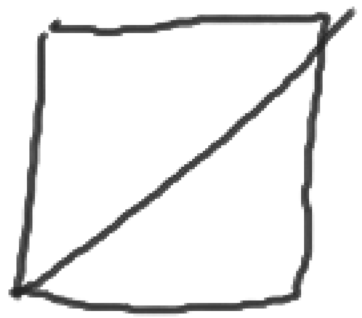
left adjoint $|\ |$ still works.

<u>A pretty fact</u>: If I have simplicial sets $S_\cdot$, $T_\cdot$, I can form a <u>product</u>: $(S_\cdot \times T_\cdot)_n = S_n \times T_n$.

<u>Proposition</u>: $|S_\cdot \times T_\cdot| \cong |S_\cdot| \times |T_\cdot|$

Δ^n diag. $\qquad \underbrace{\Delta^m \times \Delta^n}$

product of two simplices degenerate at complementary coordinates



If we are willing to consider also Δ^{op}-Top. If G is a discrete <u>abelian group</u>, then BG is again a abelian group (coordinate-wise multiplication).

$$B^n G = \underbrace{BB\cdots B}_{n} G \simeq K(n, G) \qquad \pi_k B^n G = \begin{cases} G & k = n \\ 0 & k \neq n. \end{cases}$$

Eilenberg - MacLane ~ 1950.

Wednesday: What about de Rham cohomology?