MATH 592 4/17/2024

Exam Monday

Problem ①: Homology of a 2-dimensional CW-complex 1-skeleton given by a graph, 2-cells attached by a given cycle.

② Homology via Mayer-Vietoris

③ Euler characteristic / Lefschetz fixed point theorem

④ local homology

Elementary mathematics for examples

<u>Review</u>: • tomorrow @ discussion: 1–2 PM EH 1096
• Friday in class

<u>What is next?</u> Math 695 in Fall

① "Completing the toolbox" – cohomology, homology and cohomology with coeffs.

cohomology of a space is a <u>ring</u>

First ques: tensor product

– What happens for product of spaces?

"Error terms":
= derived functors
$\otimes$, Hom $\to$ Tor, Ext

homological algebra in <u>modules over a ring</u>

Algebraic topology calculations are about "getting lucky"

The "ultimate tool of homological algebra": Spectral sequence

Where do derived functors come from? general setup: Derived categories

even for sets, spaces

homotopical algebra

"when do constructions work?"

CW-complexes "good" in some sense

what is the general picture?

→ Drop dimension axiom,

generalized homology and cohomology

"stable homotopy theory" suspension = shift? ⇒ spectra → duality

What about de Rham cohomology?

M smooth manifold — Munkres: Analysis on mflds.
(embed smoothly in $\mathbb{R}^N$, locally, graphs of smooth functions)

coordinates

cover by open subsets $\cong \mathbb{R}^n$, transition maps are diffeomorphisms.

Can define smooth functions.

Tangent vectors.

tangent space to $U \overset{open}{\subseteq} \mathbb{R}^n$ is $\mathbb{R}^n$

how do tangent vectors transform?

In coordinate point of view

$$U \xrightarrow{\cong\, f} V$$

$$U \subseteq \mathbb{R}^N \supseteq V$$

On tangent vectors: Df In coordinates, Jacobi matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$

Dual vector space to an $\mathbb{R}$-vector space V: $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) = V^\vee$

switch S,T, turn around arrows

$(?)^\vee : \mathrm{Vect}_{\mathbb{R}}^{op} \longrightarrow \mathrm{Vect}_{\mathbb{R}}$

all homomorphisms of $\mathbb{R}$-v.s.

tangent space

Differential 1-forms: Smooth sections of $(T_M)^\vee$

$M \underset{open}{\subseteq} \mathbb{R}^N$

$TM_x = \mathbb{R}^N = (\mathbb{R}^N)^\vee$

The whole trick is in coordinate changes. The space of 1-forms on M : $\Omega^1(M)$.

<u>Example</u>: $\Omega^1(U) = \{ f(x)dx \mid f : U \to \mathbb{R} \text{ smooth} \}$
$U \subseteq \mathbb{R}$ open

dual to the unit vector in $\mathbb{R}$

$U \subseteq \mathbb{R}^n$ $\sum_{i=1}^{n} f_i dx_i$ ⟵ dual to the ith coordinate vector.

- We can integrate 1-forms over curves (parametrised by smooth maps), $[0,1] \to M$

- 1-forms have better functoriality than vector fields ⟵ push forward by diffeo.

Given $f : M \to N$ smooth, we have $f^* : \Omega^1(N) \longrightarrow \Omega^1(M)$

("$dx_i = Dx_i$" - chain rule).

(not commutative)

<u>Exterior algebra</u>: $\Lambda(V)$ universal element in algebras containing V and satisfying $v.v=0$ $\forall v \in V$.

over $\mathbb{R}$:
ring with compatible scalar $\mathbb{R}$-multiplication

write $\wedge$ for .
(convention)

$$\Lambda(V) = \bigoplus_n \Lambda^n(V)$$

$\Lambda^n \mathbb{R}^N$ has basis $e_{i_1} \wedge \cdots \wedge e_{i_n}$ $\quad i_1 < \cdots < i_n$

dim $\binom{N}{n}$

$\Omega^k(M)$ = smooth sections of $\Lambda^k TM^\vee$.

k-forms

In coordinates : $\sum h_{i_1 \dots i_k} \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$

smooth function

ω k-form. $f : K \xrightarrow{\text{smooth}} M$

subset of $\mathbb{R}^k$

example: k-simplex.

$\int_{f(K)} \omega$

or K, or f

only depends on orientation of K.

Exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$

In coordinates, $d(h\, dx_{i_1} \wedge \cdots dx_{i_k}) = \sum \frac{\partial h}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

This commutes with f^* $\quad df^* = f^* d$ (chain rule)

$\Rightarrow$ well-defined on manifolds.

Verify on coordinates: $dd = 0$.

De Rham cohomology:

cochain complex

$$\left(\Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n M \right)$$

$f : M \to N$ smooth

$f^* : H^k_{DR}(N) \to H^k_{DR}(M)$

$H^k = \operatorname{Ker} d / \operatorname{Im} d$ @ Ω^k $\qquad H^k_{DR}(M)$.

<u>Theorem</u> (De Rham): $H^k_{DR}(M) = \mathrm{Hom}_{\mathbb{Z}}(H_k(M), \mathbb{R})$

homomorphisms of abelian groups.

<u>Example</u>: $H^k_{DR}(\mathbb{R}P^n) = \mathbb{R}$ $k = 0$ or $k = n$ odd

0 otherwise.

<u>Proof</u>: Stokes <u>theorem</u>: $\displaystyle\int_{\partial K} \omega = \int_K d\omega$

K k-dim,

ω $(k-1)$-form on K

$K = \Delta^k$ $\quad\quad$ $H\mathcal{C}(M) = HC^{sm}(M)$
(follows excision)

Define a pairing between the $C_k^{sm}(M)$, $\Omega^k M$ preserving d.

Use Mayer-Vietoris $\square$

Please fill out teaching evaluation.