Vector bundles and characteristic classes

(Milnor - Stasheff: Characteristic classes)

Vector n-bundle on X: open cover U_i

$U_i \times \mathbb{C}^n$ resp. $U_i \times \mathbb{R}^n$

$U_i \cap U_j \xrightarrow[\text{continuous}]{h_{ij}} GL_n(\mathbb{C})$ resp. $GL_n(\mathbb{R})$

$h_{jk} \circ h_{ij} = h_{ik}$

- Details needed: passing to a refinement of the open cover.
- Isomorphism: $\varphi_i : U_i \to GL_n(\mathbb{C})$ ... $h_{ij} \varphi_i = \varphi_j h_{ij}$.

$\check{H}^1_{\text{cont.}}(X; GL_n(\mathbb{C}))$ $\longleftarrow$ classifies vector n-bundles over $\mathbb{C}$ or $\mathbb{R}$

$GL_n(\mathbb{R})$

Another approach to vector bundles: Category Top_X : topological spaces$/X$.

Object: $Y \xrightarrow{f} X$ continuous

Morphisms:

$$\begin{tikzcd} Y \arrow[d, "\varphi"'] \arrow[dr, "f"] & \\ Y' \arrow[r, "g"'] & X \end{tikzcd}$$

Abelian group in Top_X. $\rightsquigarrow$ $\mathbb{C}$- / $\mathbb{R}$- vector space$/X$. ($=$ in Top_X).

Vector n-bundle: $\mathbb{C}$- / $\mathbb{R}$- vector space$/X$ which, when pulled back to U_i, becomes $\cong$ $\mathbb{C}^n \times U_i \to U_i$ / $\mathbb{R}^n$

Classification of vector n-bundles using homotopy classes: countable

Define $BGL_n(\mathbb{C}) \simeq BU_n$ } n-dim. vector subspaces in $\mathbb{C}^{\oplus\infty}$

$BGL_n(\mathbb{R}) \simeq BO_n$. } $\mathbb{R}^{\oplus\infty}$

We have a tautological (= universal) n-bundle

$\gamma_{\mathbb{C}}^n$ on BU_n "The fiber over a point $V \in BU_n$, $V \overset{"}{\subset} \mathbb{C}^{\oplus\infty}$ is V"

resp. $\gamma_{\mathbb{R}}^n$ on BO_n

$p : E \to BU_n$ similarly for $\mathbb{R}$

$E = \{(x,V) \mid V \in BU_n,\ x \in V\}$

Theorem: If X is paracompact, we have a bijection

$\{\cong$ classes of cx./real vector n-bundles on $X\} \cong [X, BU_n]$ / $[X, BO_n]$ unbased homotopy classes of maps

$$f^*(\gamma_{\mathbb{C}}^n) \longleftarrow\!\shortmid\ f : X \to BU_n$$

$\square$

Characteristic classes: E be a generalized cohomology theory.

$X \simeq$ CW-complex.

ξ is a real / complex n-vector bundle on X

$\longmapsto \alpha(\xi) \in E^{i}(X)$ ← only depends on $\cong$ class of ξ

E^{i}-characteristic class α on real n-bundles $\mathbb{C}$.

The only axiom: Functoriality: $f: Y \to X$ continuous map, ξ is an n-bundle on X:

$$\underset{E^{i}(Y)}{\underset{\cap}{\alpha(f^*\xi)}} = f^*(\underset{E^{i}(X)}{\underset{\cap}{\alpha(\xi)}})$$

$$E^{i}(Y) \xleftarrow{f^*} E^{i}(X)$$

The cases we will discussed at this point: real vector n-bundles: $E = \underbrace{H(?;\mathbb{Z}/2)}_{H\mathbb{Z}/2}$

Stiefel-Whitney classes

complex vector n-bundles: $E = \underbrace{H(?;\mathbb{Z})}_{H\mathbb{Z}}$

Chern classes

$E = H\mathbb{Z}[\frac{1}{2}]$ Pontryagin classes

Oriented n-bundles Euler class $H^{n}(?;\mathbb{Z})$

Computing characteristic classes:

Most fundamental case: Chern classes — via Schubert calculus

To characterise a characteristic class α, all we need is $\alpha(\gamma^n_{\mathbb{C}})$

$\alpha(\gamma^n_{\mathbb{R}})$

ξ any bundle (say, complex) on X

$f : X \to BU_n$

$$\alpha(\xi) = f^*(\alpha(\gamma^n_{\mathbb{C}}))$$

$\xi \cong f^*(\gamma^n_{\mathbb{C}})$

$\therefore$ Calculating E-characteristic classes of real (complex) n-vector bundles is equivalent to calculating $E^*(BO_n)$ $H\mathbb{Z}/2$

$E^*(BU_n)$. $H\mathbb{Z}$

CW-(co)homology: We will give BU_n (resp. BO_n) a structure of a CW-complex.

We can say something even more basic:

$$BU_1 = \mathbb{CP}^\infty \quad \text{cx. lines in } \mathbb{C}^{\oplus\infty} \qquad U_1 = S^1$$

$$BO_1 = \mathbb{RP}^\infty \qquad O_1 = \mathbb{Z}/2$$

lines in $\mathbb{R}^{\oplus\infty}$

$$H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2\,[t] \quad \text{dimension: } 1$$

$$H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}\,[z] \quad \text{dimension: } 2$$

Poincaré duality for $\mathbb{RP}^n$, $\mathbb{CP}^n$

First thing next time: An easy guess of what might, most naively, happen for $n > 1$