# Math 696

Friday, January 10, 2025 8:59 AM

? $H^*(BU(n);\mathbb{Z})$

$H^*(BO(n);\mathbb{Z}/2)$

$BU(n) \simeq BGL_n(\mathbb{C})$

$BO(n) \simeq BGL_n(\mathbb{R})$

$$\underbrace{S^1\times\cdots\times S^1}_{n} = U(1)\times\cdots\times U(1) \subseteq U(n)$$

$$BU(1)\times\cdots\times BU(1) \xrightarrow{\oplus} BU(n)$$

$$\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty$$

lines (through origin) in $\mathbb{C}^{\oplus\infty}$

n-dim v. subspaces of $\mathbb{C}^{\oplus\infty}$

$\infty = n\infty$

In cohomology:

$H^*\mathbb{CP}^\infty = \mathbb{Z}[z]$ dim. 2

$$H^*BU(n) \xrightarrow{\varphi_{\mathbb{C}}} H^*(BU(1)\times\cdots\times BU(1)) = \mathbb{Z}[z_1,\dots,z_n]$$

permutation action $\circlearrowleft$

$$H^*BU(n) \to H^*(BU(1)\times\cdots\times BU(1))^{\Sigma_n} = \mathbb{Z}[z_1,\dots,z_n]^{\Sigma_n}$$

Similarly, $H^*(BO(n);\mathbb{Z}/2) \xrightarrow[\varphi_{\mathbb{R}}]{} H^*((\mathbb{RP}^\infty)^n;\mathbb{Z}/2)^{\Sigma_n} = \mathbb{Z}/2[t_1,\dots,t_n]^{\Sigma_n}$

dim. 1

**Theorem**: The maps $\varphi_{\mathbb{C}}, \varphi_{\mathbb{R}}$ are isomorphisms of rings.

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**Lemma:**
R commutative ring

symmetric group acts by permutation

$$R[z_1,\dots z_n]^{\Sigma_n} = R[\sigma_1,\dots \sigma_n]$$

$$\sigma_k = \sum_{i_1<\dots<i_k} z_{i_1}\cdots z_{i_k}$$ ← elementary symm. polynomials

∴ Theorem implies:

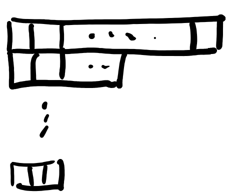
$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1,\dots c_n]$ $\qquad c_k = \sigma_k(z_1,\dots z_n) \in H^*((\mathbb{CP}^\infty)^n)$ — Chern classes

$H^*(BO(n);\mathbb{Z}/2) = \mathbb{Z}/2[w_1,\dots w_n]$ $\qquad w_k = \sigma_k(t_1,\dots t_n) \in H^*((\mathbb{RP}^\infty)^n;\mathbb{Z}/2)$ — Stiefel-Whitney classes

Proof of lemma: $\Sigma_n$ acts by permutation on monomials.

$R[z_1,\dots z_n]^{\Sigma_n}$ is a free R-module on $\displaystyle\sum_{\substack{\text{all permutations}\\ \text{of } (l_1,\dots l_n)}} z_1^{l_1}\cdots z_n^{l_n} = \sigma(l_1,\dots l_n)$

$l_1 \geq \dots \geq l_n \geq 0$:



$l_1$ boxes
$l_2$ boxes
⋮
$l_n$ boxes

} Young diagram $(l_1,\dots l_n)$

Understanding product of Young diagrams
symmetrization of monomials

Lexicographic ordering:
$(l_1 \geq \dots \geq l_n) > (m_1 \geq \dots \geq m_n)$

$l_1 = m_1$
⋮
$l_{j-1} = m_{j-1}$
$l_j > m_j$

well linearly ordered commutative monoid
(+ coordinate-wise)

We have an increasing filtration on symmetric polynomials

$F_\lambda = \langle$ all symmetrized monomials corresponding to Young diagrams $\mu < \lambda \rangle$

Young diagram

to Young diagrams $\mu \subset \lambda \rangle$

Young diagram

$$F_{\lambda_1} \cdot F_{\lambda_2} \subseteq F_{\lambda_1 + \lambda_2}$$

<u>Associated graded object</u> with respect to this filtration:

$$\sigma(l_1, \dots l_n) \cdot \sigma(m_1, \dots m_n) = \sigma(l_1 + m_1, \dots l_n + m_n)$$

In the actual $R[z_1, \dots z_n]^{S_n}$
there are terms in lower filtrations.

The associated graded object is easily seen to be a polynomial ring $R[X_1, \dots X_n]$ $\quad X_i = z_1 \cdots z_i$

the power of $x_i$ is $l_i - l_{i-1}$.

A polynomial R-algebra is a free commutative R-algebra.
So if an associated graded object of an R-algebra A is polynomial, $R[x_1, \dots x_n]$
then so is A.

$$R[x_1, \dots x_n] \longrightarrow A$$
$$x_i \longmapsto \text{any representative}$$

homomorphism of commutative R-algebras, filtered, induces $\cong$ on associated graded.

Grading well-ordered. By (transfinite) induction, $\cong$. $\square$

Let us discuss a proof of the Theorem. We give a CW-decomposition of $BU(n)$. Describing the open cells:

$$V \subset \mathbb{C}^{\oplus\infty} \quad n\text{-dim vector subspace.}$$

$$V = \langle v_1, \dots, v_n \rangle$$

basis

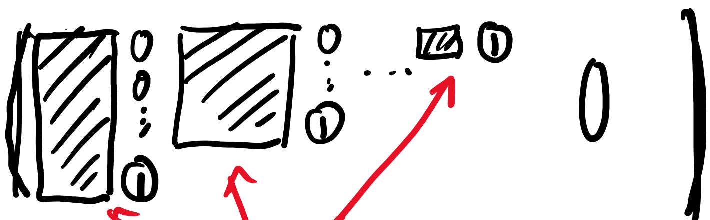
row vectors (by convention)

Base change = row operations.

$$\begin{pmatrix} v_1 & & \\ \vdots & & \\ v_n & & \end{pmatrix}$$

$n \times \infty$ matrix (each row finitely many $\neq 0$ entries)

Reduced row echelon form. (By convention, in this context, we reverse the columns.)



← RREF

shaded areas are arbitrary $\in \mathbb{C}$

$\{$ all $V \subset \mathbb{C}^{\oplus\infty}$, $\dim V = n$ with given RREF $\}$ $\cong$ $\{$ all choices of shaded entries $\}$

$\cong \mathbb{C}^N$ for some $N$

as a topological space ← even-dimensional Euclidean space

<u>Schubert cell</u>