Math 696

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$BU(n)$ = n-dim. vector subspaces $\subseteq \mathbb{C}^{\oplus\infty}$ ← ω

real analogue also

(Open) Schubert cells: $\langle \underbrace{v_1, \dots v_n}_{\text{row vectors}} \rangle$

RREF:
n pivot

$$\left(\begin{array}{ccccccc} & 0 & \boxed{**} & 0 & \boxed{*} & 1 & \\ \cdots & 0 & & 1 & & 0 & 0 \\ & 1 & & 0 & & \vdots & \\ & 0 & & \vdots & & 0 & \\ & 0 & & 0 & & & \\ & \vdots & & & & & \\ & 0 & & & & & \end{array}\right)$$

← direction of pivots

arbitrary matrices of complex numbers

To get an actual CW-decomposition: we need closed (compact) cells

$$\begin{array}{ccc} D^N & \longrightarrow & BU(n) \\ \cup & & \cup \\ S^{N-1} & \longrightarrow & BU(n)_{N-1} \end{array}$$

attaching map

unitary (real case: orthogonal)

Variant of RREF: UREF

$$\left(\begin{array}{ccccccc} \cdots & \vdots & \boxed{**} & \vdots & \boxed{*} & p_1 & \\ & p_3 & & p_2 & & & 0 \end{array}\right)$$

pivots $p_k \in \mathbb{R}_{>0}$
rows are orthonormal.

Gram-Schmidt process $\Rightarrow$ UREF also unique $\longleftrightarrow$ RREF

UREF cells extend properly to the boundary.

The space we get for closed cells:

$$0 < k_1 < \cdots < k_n$$

Data: $v_i \in D^{2k_i}$ all orthogonal (using embedding $\mathbb{C}^{k_i} \subseteq \mathbb{C}^{k_j}$ by initial coordinates)

unit disk in $\mathbb{C}^{k_i}$

k_i'th coordinate $\mathbb{R}_{\geq 0}$

In detail: Milnor-Stasheff

iterated disk fiber bundles, but fiber bundles over a disk (contractible paracompact space) are trivial.

Now we know that the (co)homology $H^N(BU(n))$ has the rank of the number of Young diagrams with N cells $=$ # of symmetric monomials in n variables of degree N. How do we prove that

$$H^N(BU(n)) = H^N(\underbrace{BU(1) \times \cdots \times BU(1)}_{n})^{\Sigma_N} \text{ ?}$$

$$BU(1) \times \cdots \times BU(1) \xrightarrow[\oplus]{} BU(n)$$

$BU(1) = \mathbb{C}P^\infty$

if we had a cell map (preserving cell filtration) we could just read it off on cells

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The method to treat this: Schubert calculus. ← reductive algebraic groups / $\mathbb{C}$

The case of $GL_n(\mathbb{C})$: Flag varieties

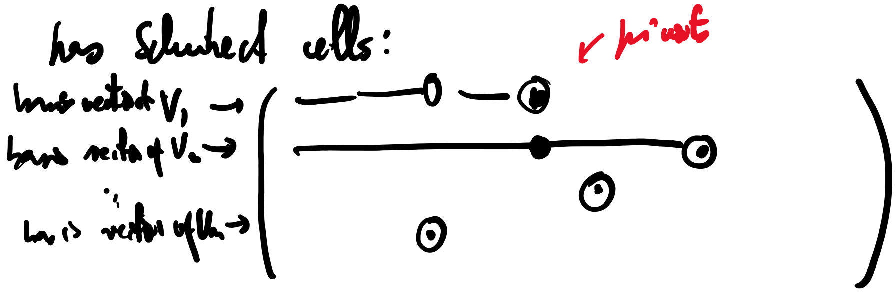
$$\{(V_1 \subset \dots \subset V_\ell) \mid \text{Vector subspaces of } \mathbb{C}^M,\ \dim V_i = k_i\}$$

colim $M \to \infty$

The case we are interested in:

$$X = \{(V_1 \subset \dots \subset V_n) \mid \text{vector subspaces of } \mathbb{C}^{\oplus\infty},\ \dim V_i = i\} \simeq \underbrace{\mathbb{CP}^\infty \times \dots \times \mathbb{CP}^\infty}_{n}$$

Then X is a Schubert variety, so it also has Schubert cells:



Variant of RREF same as for $BU(n)$, but pivots are permuted.

In the original RREF, we eliminate the numbers above a pivot. In the new variant, this only occurs for previous pivots to the right of the given pivot. So we have a cell map $X \to BU(n)$ but the corresponding cell in X has a higher dimension except when the X-pivots are already ordered right to left. But cells in X are generators of (co)homology of X, so $(H_* X)_{\Sigma_n} \xrightarrow{\cong} H_*(BU(n))$ dualize $\Rightarrow$ cohomology $\square$

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So to recap, we have proved : $H^*(BU(n);\mathbb{Z}) = \underbrace{H^*((\mathbb{C}P^\infty)^n;\mathbb{Z})^{\Sigma_n}}_{\mathbb{Z}[z_1,\dots z_n]^{\Sigma_n}}$

$= \mathbb{Z}[c_1,\dots c_n]$

chern classes

$c_i = \sigma_i(z_1,\dots z_n)$

$|z_i| = 2$

Analogously, $H^*(BO(n);\mathbb{Z}/2) = H^*((\mathbb{R}P^\infty)^n;\mathbb{Z}/2)^{\Sigma_n} \leftarrow \mathbb{Z}/2[t_1,\dots t_n]^{\Sigma_n}$

$= \mathbb{Z}/2[w_1,\dots w_n]$

$w_i = \sigma_i(t_1,\dots t_n)$

$|t_i| = 1$

why $d^{cell} = 0$? cells are no longer in even dimensions, Because it is true for $\mathbb{R}P^\infty$.

Formulas for $c_k(\xi \oplus \eta)$, $w_k(\xi \oplus \eta)$

dim n_1, n_2 (complex bundles); dim n_1, n_2 (real bundle)

$z_1, \dots z_{n_1}$

$z_{n_1+1}, \dots z_{n_1+n_2}$ $\quad n = n_1 + n_2$

$$\sigma_k = \sum_{i+j=k} \sigma_i \cdot \sigma_j$$

understood that $\sigma_0 = 1$

Whitney formula:

$$c_k(\xi \oplus \eta) = \sum_{i=0}^{k} c_i(\xi)\, c_{k-i}(\eta)$$

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi)\, w_{k-i}(\eta)$$

Application :

Recall the Whitney immersion theorem:
Every smooth (compact) manifold M of dim. $n > 1$ has an immersion $\subset \mathbb{R}^{2n-1}$.

(Immersion $\mathbb{RP}^2 \subset \mathbb{R}^3$ is called the Boy surface)

$\mathbb{RP}^{2^n}$ **cannot** be immersed into $\mathbb{RP}^N$ for $N < 2^{n+1} - 1$.

Stiefel-Whitney classes