Whitney formula:

$$w_n(\xi \oplus \eta) = \sum_{k=0}^{n} w_k(\xi)\, w_{n-k}(\eta) \qquad \big(w_0(\xi) = 1\big)$$

Write formally: $w(\xi) = 1 + w_1(\xi) + \cdots + w_n(\xi) \in H^*(X)$
for a real n-bundle ξ:

ξ real n-bundle on X

$$w(\xi \oplus \eta) = w(\xi)\, w(\eta)$$

$w(1) = 1$

trivial 1-dim. real bundle

$\therefore$ Stiefel-Whitney (also Chern) classes make sense on virtual bundles.

<u>Application</u>: Example accompanying the Whitney immersion theorem:

M is an n-manifold, $n > 1$, $\exists$ immersion $M \to \mathbb{R}^{2n-1}$.

smooth injective on tangent space at each point

We can show that $\mathbb{R}P^{2^n} \not\xrightarrow{\text{immersion}} \mathbb{R}^k$ $\quad k < 2^{n+1} - 1$

Study $\tau_{\mathbb{RP}^n} \oplus 1 \cong (n+1)\gamma^1$. γ^1 = tautological real line bundle on $\mathbb{RP}^n$

$$\mathbb{RP}^n = S^n/(\mathbb{Z}/2) \qquad S^n \subset \mathbb{R}^{n+1}$$

unit sphere

$$\tau_{S^n} \oplus 1 = (n+1)1$$

normal bundle

$\mathbb{Z}/2$-equiv.

take space of orbits

$$\tau_{\mathbb{RP}^n} \oplus 1 = (n+1)\gamma^1$$

If ξ is an n-bundle, $w_k\xi = 0$ $k > n$.

$$H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[t] \qquad H^*(\mathbb{RP}^n; \mathbb{Z}/2) = \mathbb{Z}/2[t]/(t^{n+1})$$

in dim 1

$$t = w_1(\gamma^1)$$

Whitney formula

$$w(\gamma^1) = 1+t \qquad w(\tau_{\mathbb{RP}^n}) = (1+t)^{n+1} \in \mathbb{Z}/2[t]/(t^{n+1})$$

$$n = 2^k \qquad w(\tau_{\mathbb{RP}^{2^k}}) = (1+t)^{2^k+1} = (1+t^{2^k})(1+t) =$$
$$= 1+t+t^{2^k} \qquad (t^{2^k+1}=0)$$

assume we have an immersion $\tau_{\mathbb{RP}^{2^k}} \longrightarrow \mathbb{R}^N$

we get a normal bundle ν

$$\tau_{\mathbb{RP}^{2^k}} \oplus \nu = N$$

$$w(\tau_{\mathbb{RP}^{2^k}})\,w(\nu) = 1 \Rightarrow w(\nu) = \frac{1}{w(\tau_{\mathbb{RP}^{2^k}})} = \frac{1}{1+t+t^{2^k}} = (1+t^{2^k})^{-1}(1+t)^{-1} =$$
$$= ((1+t^{2^k})(1+t+\dots+t^{2^k})$$
$$= 1+t+t^2+\dots+t^{2^k-1}$$

$\therefore \dim \nu \geq 2^k - 1$

Wednesday, January 15, 2025 9:21 AM

Yet another take on the Whitney formula: (parallel in real and cx. case)

coeff. $\mathbb{Z}/2$ (real) / coeff. $\mathbb{Z}$ (cx. case)

Discussing cx. case:

$$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1, \dots c_n]$$

There is no reason for $n>1$ why $H_* BU(n)$ would be a ring.

The Whitney formula is related to the embedding $BU(k) \times BU(n-k) \overset{\oplus}{\hookrightarrow} BU(n)$

Take $\operatorname{colim}_{k,n\to\infty}$.

$$BU \times BU \xrightarrow{\oplus} BU.$$

commutative, associative, unital, (additive) inverse up to $\simeq$

Now we have

$$H^*(BU;\mathbb{Z}) = \mathbb{Z}[c_1, c_2, c_3, c_4, \dots\dots]$$

$H_*(BU(n);\mathbb{Z})$ is also a comm. ring (using Künneth formula)

But let's stay in cohomology

$$\oplus^* = \psi : H^*(BU;\mathbb{Z}) \to H^*(BU;\mathbb{Z}) \otimes H^*(BU;\mathbb{Z})$$

<u>Homomorphism of rings</u>

$$\psi(c_n) = \sum_{k=0}^{n} c_k \otimes c_{n-k}$$

terminology varies, in some cases we ask for maps modelling the inverse (conjugation)

In rough terms, an algebra A with a coproduct $\psi : A \to A \otimes A$ which is a homomorphism of algebras, ψ co-commutative, co-associative co-unital is called a <u>Hopf Algebra</u>

If we use the above definition (without inverse), if we have a Hopf algebra which is a free module over its grounded ring, we can take its dual.

also a Hopf algebra

$\psi \leftrightarrow$ product

J. Peter May : Hopf Algebras (his web page)

Say, we are graded by $\mathbb{N}_0$, in each degree, the module is finitely generated free.

Dualize degree by degree.

$H^* BU = \mathbb{Z}[c_1, c_2, \dots]$ is a Hopf algebra,

$$\psi(c_n) = \sum_{k=0}^{n} c_k \otimes c_{n-k}$$

What does its dual look like? We could do this algebraically, but topology helps: The dual is $H_* BU \supseteq H_* \mathbb{C}P^\infty = \mathbb{Z}\{1, b_1, b_2, \dots\}$

free ab. group

$\dim b_i = 2i$

The considerations of the Schubert cells give:

$$H_* BU = \mathbb{Z}[b_1, b_2, \dots] = \operatorname{colim}_{n \to \infty} \left((\mathbb{Z}\{1, b_1, b_2, \dots\})^{\otimes n} \right)^{\Sigma_n}$$

Proposition: $H^* BU$, $H_* BU$ are dual bi-polynomial Hopf algebras.

itself and its dual are both polynomial

What's next: Stiefel-Whitney / Chern (Pontryagin) <u>numbers</u>

M is a compact n-manifold

real / complex

$i_1 + \dots + i_k = n$

$w_{i_1} \cdots w_{i_k}[M] = \langle w_{i_1} \cdots w_{i_k}(\tau_M), [M] \rangle \in \mathbb{Z}/2$

Kronecker pairing

Poincaré duality fundamental class in $H_n(M; \mathbb{Z}/2)$

$c_{i_1} \cdots c_{i_k}[M] = \langle c_{i_1} \cdots c_{i_k}[\tau_M], [M] \rangle \in \mathbb{Z}$

$\in H_{2n}(M; \mathbb{Z})$

The Stiefel-Whitney resp. Chern numbers are invariants under unoriented resp. complex <u>cobordism</u>.