Math 696

Friday, January 17, 2025 8:57 AM

$M$ compact $n$-manifold

$$m_1 + 2m_2 + \cdots + km_k = n$$

Then we have a Stiefel-Whitney number:

$$\mathbb{Z}/2 \ni w_1^{m_1} w_2^{m_2} \cdots w_k^{m_k}[M] = \langle w_1^{m_1} \cdots w_k^{m_k}(\tau_M), [M] \rangle$$

fundamental class $\in H_n(M; \mathbb{Z}/2)$

Kronecker pairing

Instead of $\tau_M$, we can also consider the virtual normal bundle $\nu_M$: $\tau_M \oplus \nu_M = 0$

$$w(\nu_M) = w(\tau_M)^{-1}$$

(Unoriented) cobordism: $M_1 \sim M_2$ when $M_1 \amalg M_2 = \partial N$

$\mathbb{Z}/2$-module $\Omega_n$ — $N$ compact manifold with boundary

addition: $[M_1] + [M_2] = [M_1 \amalg M_2]$

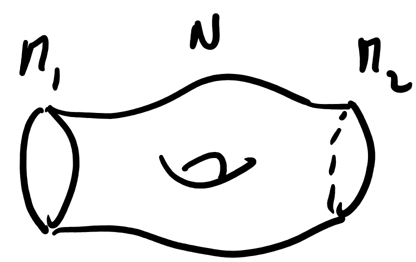
$2[M] = 0$ $\quad M \amalg M = \partial M \times [0,1]$

<u>Proposition</u>: If $M_1, M_2$ are cobordant, then

$$w_{m_1,\ldots,m_k}[M_1] = w_{m_1,\ldots,m_k}[M_2].$$

Proof:

$M_1 \amalg M_2 = \partial N$ $\qquad i: M_1 \amalg M_2 \hookrightarrow N$




homological class

$$\langle i^* \alpha, b \rangle = \langle \alpha, i_* b \rangle$$

$$i_* [\underbrace{M_1 \amalg M_2}_{\partial[N]}] = 0.$$

$$\tau_{M_1 \amalg M_2} \oplus 1 \cong \tau_N \big|_{M_1 \amalg M_2}$$

$$w_1^{m_1} \cdots w_k^{m_k}(\tau_{M_1 \amalg M_2}) = i^* w_1^{m_1} \cdots w_k^{m_k}(\tau_N)$$

$\square$

**Theorem**: $w_1^{n_1} \cdots w_k^{n_k}[M]$, $n_1 + 2n_2 + \cdots + kn_k = n$,
$n_i$ are not of the form $\underbrace{2^\ell - 1}$ is a complete system of invariant for the unoriented cobordism group $\Omega_n$.

**Example**: $\Omega_0 = \mathbb{Z}/2$
$\Omega_1 = 0$
$\Omega_2 = \mathbb{Z}/2$ $\quad w_2[M]$
$\Omega_3 = 0$ $\quad \mathbb{RP}^2 \times \mathbb{RP}^2, \mathbb{RP}^4$
$\Omega_4 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ $\quad w_2^2, w_4$

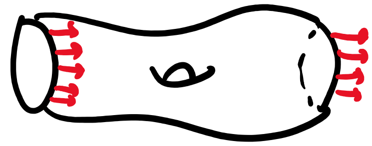
$$H^*(\mathbb{RP}^2; \mathbb{Z}/2) = \mathbb{Z}/2[\underset{\substack{\uparrow \\ \deg = 1}}{t}]/t^3$$

$M = \mathbb{RP}^2$
$w_2[M] =$
$= \langle t^2, [M] \rangle = 1.$ ✓

$w(T_{\mathbb{RP}^2}) = (1+t)^3 =$
$= 1 + t + t^2$

Oriented cobordism : Oriented compact manifolds

$M_1 \sim M_2$



$M_1 \sqcup M_2 = \partial N$

$N$ oriented $(n+1)$-manifold with $\partial$

$$\tau_N \big|_{M_i} \cong \tau_{M_i} \oplus 1$$

Require that this preserves orientation with the 1-summand oriented inward for $i=0$, outward for $i=1$

Group structure $[M_1] + [M_2] = [M_1 \sqcup M_2]$ $-[M]$ is $M$ with reversed orientation

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Perhaps better replace $\tau_M$ with $\nu_M$

this can be thought of as $\tau_M \oplus \nu_M^N \cong N$
actual bundle

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Analogous theorem: we can obtain a complete system of invariants of $\Omega_n^{SO}$ using Stiefel-Whitney numbers and Pontryagin numbers ← to be covered

# A more fundamental case for homotopy theory: Complex Cobordism.

**Stably weakly complex manifolds**: Compact real n-manifold $M$

$\nu_M^N$ ← number: $N-n$ even

$\nu_M^N \oplus \tau_M \xrightarrow{\cong} N$

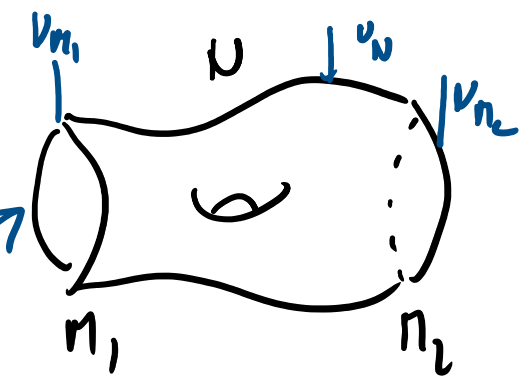
give $\nu_M^N$ a structure of a complex bundle

} data

vary $n$: $N \mapsto N \oplus 2$

"standard" cx. structure

pick: $\mathbb{R}^2 \cong \mathbb{C}$




We can say $M_1 \sim_{cx} M_2$ if

$M_1 \amalg M_2 = \partial N$ ← compact stably weakly cx. mfld.

$\nu_N|_{M_i} \overset{\text{and}}{\cong} \nu_{M_i}^N$ as cx. bundle

opposite orientation of the 1

We have Chern numbers: $2n_1 + 2\cdot 2 n_2 + \dots + 2\cdot k\, n_k = n$

$$c_1^{n_1}\cdots c_k^{n_k}[M] = \langle c_1^{n_1}\cdots c_k^{n_k}(\nu_M), [M]\rangle \in \mathbb{Z}$$

} only non-trivial for $n$ even

Using the Hopf algebra structure on $H^*(BU;\mathbb{Z})$:

dual $\rightleftarrows$ $H_*$

All the Chern numbers can also be thought of as a homomorphism (preserving dimension degree)

dual Hopf algebra

$$\Omega^{U} \longrightarrow (H^*(BU;\mathbb{Z}))^\vee = H_*(BU;\mathbb{Z})$$

complex cobordism ring

$[M_1]\cdot[M_2] = [M_1 \times M_2]$

Whitney formula $\Rightarrow$ homomorphism of rings

$\mathbb{CP}^\infty \longrightarrow BU$

$\mathbb{CP}^\infty = BU(1)$

$H_*(\mathbb{CP}^\infty;\mathbb{Z}) \to H_*(BU;\mathbb{Z})$

$H_*(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}\{b_0, b_1, b_2, \dots\}$

$H_*(BU;\mathbb{Z}) = \mathbb{Z}[b_1, b_2, \dots]$

$b_0 = 1$

$|x_i| = 2i$

So the Chern number defines a homomorphism of rings

$$\Omega^U \xrightarrow{\ \varphi\ } \mathbb{Z}[b_1, b_2, \dots] = H_*(BU;\mathbb{Z})$$

complex cobordism ring

linear combinations of monomials of algebraic degree $> 1$

<u>Theorem</u>: The homomorphism $\varphi$ is injective. Its image is a polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$, $|x_i| = 2i$. Modulo decomposables,

$$x_i \equiv b_i \quad \text{if } i \text{ is not of the form } p^l - 1,\ p \text{ prime}$$
$$x_i \equiv p\, b_i \quad \text{if } i = p^l - 1,\ p \text{ prime.} \quad \square$$

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What is this ring $\mathbb{Z}[x_1, x_2, \dots]$? This is the <u>Lazard ring</u>, the universal ring of formal group laws.