E is an $\underbrace{\text{associative commutative unital}}_{\text{ACU}}$ ring spectrum (generalized cohomology theory)

we say that E is complex-oriented when the universal complex line bundle $\gamma^1_{\mathbb{C}}$ on $\mathbb{C}P^\infty$ is E-oriented.

Review: If ξ is a vector real n-bundle on X, then the Thom space X^ξ is the 1-point compactification of the total space E of ξ if X is compact, $\operatorname{colim}_{\substack{K \subset X \\ \text{compact}}} K^\xi$ in general. We have the Thom diagonal

$X_+ = X \amalg \{*\}$

$$X^\xi \xrightarrow{\theta} X^\xi \wedge X_+ \qquad p : E \to X.$$

$$x \in E \longmapsto (x, p(x))$$

From this, we get a pairing

$$\theta^* : E^* X \otimes \tilde{E}^* X^\xi \longrightarrow \tilde{E}^* X^\xi$$

We say that ξ (real n-bundle) is E-oriented if there exists a class $\underbrace{u \in \tilde{E}^n X^\xi}_{\text{Thom class}}$ which, for $x \in X$, restricts to a unit:

$$\tilde{E}^n X^\xi \longrightarrow \tilde{E}^n \{x\}^\xi = \tilde{E}^n S^n = E^0(*).$$

$$u \longmapsto \text{invertible element } (= \text{unit})$$

Thom isomorphism theorem: Pairing with a Thom class (if one exists) is an isomorphism (the Thom isomorphism):

$$E^k X \otimes \tilde{E}^n X^{\xi} \xrightarrow{\theta_j^*} \tilde{E}^{n+k} X^{\xi}$$

$$\theta_j^*(? \otimes u) : E^k X \xrightarrow{\cong} \tilde{E}^{n+k} X^{\xi}$$

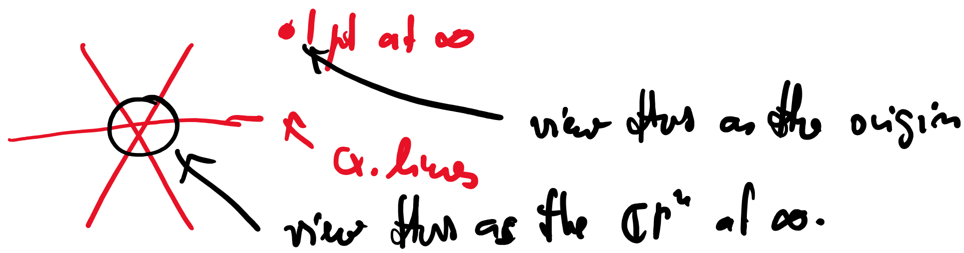
reduced suspension.

("To the eyes of E, the Thom space X^{ξ} just looks like $\Sigma^n X_+$.") $\square$

<u>Comment</u>: If we have a Thom class $u \in \tilde{E}^n X^{\xi}$, its pullback under the 0-section map $X_+ \to X^{\xi}$ is called the <u>Euler class</u> $e(\xi) \in E^n(X)$.

What is the Thom space $(\mathbb{CP}^{\infty})^{\gamma^1_{\mathbb{C}}}$ of the universal complex line bundle? It is $\mathbb{CP}^{\infty}$ again.

$$(\mathbb{CP}^n)^{\gamma^1} \cong \mathbb{CP}^{n+1}.$$



∴ If E is a complex-oriented generalized cohomology,

$$E^*\mathbb{CP}^n = E^*[u]/u^{n+1}$$

a class in E^2; the Thom class of γ^1.

$$E^*\mathbb{CP}^{\infty} = E^*[[u]]$$

there could be infinite series in u to add in the same degree (don't care about different degrees)

Example: $E = K$ (complex K-theory)

$$E_* = E_*(*) = \mathbb{Z}[\beta, \beta^{-1}]$$ β is the **Bott class**

homological degree 2

$$\beta \in E^{-2}(*) = E_2(*)$$ cohomological degree -2.

K is a complex-oriented theory: the Euler class of γ^1: $\gamma^1 - 1$.

$$K^* \mathbb{C}P^\infty = \mathbb{Z}[\beta, \beta^{-1}][[u]]$$

cohomolog. degree -2 cohomolog. degree 2

Example of a legal element of $K^0 \mathbb{C}P^\infty = 1 + \beta u + \beta^2 u^2 + \dots$

Side note: The Schubert calculus proof we did any complex-oriented generalized cohomology theory:

(Atiyah-Hirzebruch spectral sequence)

$$H^p(X, E^q) \Rightarrow E^{p+q}(X)$$

collapses

$$E^* BU(n) = E^*[[u_1, \dots, u_n]]^{S_n}$$

$$= E^*[[c_1, \dots c_n]]$$

E-valued Chern classes

All complex vector bundles are E-oriented.

How do formal group laws appear? Let E^* be a complex-oriented generalized cohomology theory.

$$(*)\quad \begin{cases} E^*\mathbb{C}P^\infty = E^*[[u]] \\ E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E^*[[x, y]] \end{cases}$$

u_1, u_2

We actually have a map

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\otimes} \mathbb{C}P^\infty$$

classifying the tensor product of complex line bundles.

in $H^2(?;\mathbb{Z})$ $\quad \mathbb{C}P^\infty = K(\mathbb{Z},2) \qquad K(\mathbb{Z},2) \times K(\mathbb{Z},2) \longrightarrow K(\mathbb{Z},2)$

line bundles $\xrightarrow{c_1}$ $\qquad [X, K(\mathbb{Z},2)] = H^2(X,\mathbb{Z}) \qquad$ + in H^2

$$c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$$

$(*)$:

$$E^*\mathbb{C}P^\infty \xrightarrow{E^*(\otimes)} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

$$E^*[[u]] \longrightarrow E^*[[x,y]]$$

$$u \longmapsto F(x,y)$$

$$\therefore\ c_1^E(\xi \otimes \eta) = F(c_1^E(\xi), c_1^E(\eta))$$

complex line bundles

What kind of power series is F?

Answer: <u>A formal group law</u> (FGL)

A **formal group law (FGL)** over a commutative ring R is a power series $F(x,y) \in R[[x,y]]$ which satisfies:

$\underset{\text{\ensuremath{\mathbin{/\!/}}}}{}$ $E^{even}(*)$

$$F(x,0) = F(0,x) = x$$
$$F(x,y) = F(y,x)$$
$$F(F(x,y),z) = F(x, F(y,z)).$$

If E is a complex-oriented generalized cohomology theory, we may get different FGL.

Examples: $E = H^*(?;\mathbb{Z})$ $\quad F(x,y) = x+y$ Additive FGL

In general, we write $F(x,y) = x +_F y$

In K-theory, multiplication is the tensor product.

$1+\beta u$ $\qquad 1+\beta(x+_F y) = (1+\beta x)(1+\beta y)$ $\qquad \beta$ = Bott periodicity

$$x +_F y = x + y + \beta xy$$ ← Multiplicative FGL

What about complex cobordism? : (we will prove it is complex-oriented)

The FGL of complex cobordism is universal.

Construct the universal FGL:

$$F(x,y) = \sum a_{ij} x^i y^j$$

$$L = \mathbb{Z}[a_{ij} \mid i,j \in \mathbb{N}_0] \,/\, \text{relations enforcing that } F \text{ is an FGL on } L.$$

$a_{01} = 1 \quad a_{0k} = 0 \quad k \neq 1$

$a_{ij} = a_{ji}$

associativity

L is called the Lazard ring.

$$\{\text{FGL on a ring } R\} \cong \mathrm{Mor}_{\text{Rings}}(L, R)$$

Punchline: $\Omega^{\mathbb{C}}_{*} = L$.

$\Omega^{\mathbb{C}}_{*}$: complex cobordism ring