Math 696

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More details on FGL of K-theory

Bott class $\beta \in \tilde{K}^0 \underbrace{S^2}_{\mathbb{C}P^1}$

$= \gamma^1_{\mathbb{C}} - 1$ on $\mathbb{C}P^\infty$, can restrict to $\mathbb{C}P^1$

this class exists in $\tilde{K}^0 \mathbb{C}P^\infty \ni u$

Atiyah-Hirzebruch spectral sequence

$$\underbrace{H^p(\mathbb{C}P^\infty, K^q(*))}_{\mathbb{Z}[u]\ \ \mathbb{Z}[\beta,\beta^{-1}]} \Rightarrow K^{p+q}\mathbb{C}P^\infty \leftarrow \mathbb{Z}[[u]][\beta,\beta^{-1}]$$

$\underbrace{\mathbb{Z}[u]\qquad\qquad}_{\mathbb{Z}[u,\beta,\beta^{-1}]}$

permanent cycles, also permanent cycle: $\gamma^1_{\mathbb{C}} - 1$ represented by

Formal group law: $c_1(\xi\eta) = c_1(\xi) +_F c_1(\eta) = F(c_1(\xi), c_1(\eta))$ ← formal group law

ξ, η cx. line bundles

$c_1(\gamma^1) = u = \gamma^1 - 1 \in K^0(\mathbb{CP}^\infty)$ (omitting the Bott periodicity β)

$c_1(\xi) = \xi - 1 \in K^0(X)$

any cx. line bundle on X

$$c_1(\xi\eta) = \underbrace{c_1(\xi)}_{x} +_F \underbrace{c_1(\eta)}_{y}$$

$$\left.\begin{aligned} \xi &= x+1 \\ \eta &= y+1 \end{aligned}\right\} \quad \begin{aligned} \xi\eta &= (x+1)(y+1) \\ c_1(\xi\eta) &= (x+1)(y+1) - 1 \\ &= x + y + xy \end{aligned}$$

<u>multiplicative FGL</u>

<u>Example</u>: let ξ be a vector bundle on X (CW-complex)

$S(\xi)$ = unit sphere bundle (using Euclidean metric)

$\cap$

$D(\xi)$ = unit disk bundle

$\downarrow$

X

So we have a (based) cofibration sequence:

$$S(\xi)_+ \longrightarrow D(\xi)_+ \longrightarrow X^\xi$$

$$\boxed{S(\xi)_+ \longrightarrow X_+ \longrightarrow X^\xi}$$

The Gysin sequence

In our example, let us look at $(\gamma^1_{\mathbb{C}})^2$

$$S(\gamma^1_{\mathbb{C}}) \simeq * = S(\mathbb{C}^{\oplus\infty})$$

$$S((\gamma^1_{\mathbb{C}})^2) = S(\mathbb{C}^{\oplus\infty})/x\sim -x = \mathbb{RP}^\infty$$

same for $\mathbb{R}^{\oplus\infty}$

So the Gysin sequence:

$$\mathbb{RP}^\infty_+ \to \mathbb{CP}^\infty_+ \to (\mathbb{CP}^\infty)^{(\gamma^1_{\mathbb{C}})^2}$$

Thom space (= mapping cone)

We can use this to calculate $K^*(\mathbb{RP}^\infty)$.

$$K^1\cdots \leftarrow \underbrace{K^0(\mathbb{RP}^\infty)}_{?} \leftarrow K^0\mathbb{CP}^\infty \leftarrow \tilde{K}^0(\mathbb{CP}^\infty)^{(\gamma^1_{\mathbb{C}})^2} \leftarrow K^{-1}\mathbb{RP}^\infty$$

$= K^1\mathbb{RP}^\infty$

$$\mathbb{Z}[[u]] \xleftarrow{c_1((\gamma^1_{\mathbb{C}})^2)} \mathbb{Z}[[u]]$$

we have the Thom isomorphism (applying Bott periodicity)

$$c_1(\gamma^1_{\mathbb{C}}) +_F c_1(\gamma^1_{\mathbb{C}}) = u +_F u = [2]_F u = 2u + u^2$$

non-zero divisor

$$K^0(\mathbb{RP}^\infty) = \mathbb{Z}[[u]]/(2u+u^2)$$

$$K^1(\mathbb{RP}^\infty) = 0$$

$(2+u)u$

Additively: $K^0\mathbb{RP}^\infty = \mathbb{Z} \oplus \mathbb{Z}_2$.

$$\mathbb{Z}[[u]]/(2+u) = \mathbb{Z}_2$$

Compare: $H^*(\mathbb{RP}^\infty; \mathbb{Z}) = \mathbb{Z}[u]/2u$

Finite group G (compact Lie)
Atiyah-Segal:
$$K^0BG = (RG)^\wedge_I$$
$$K^1BG = 0$$

<u>Exercise</u>: ① As a ring, $K^0\mathbb{RP}^\infty = \left(\mathbb{Z}[\alpha]/(\alpha^2-1)\right)^\wedge_{(\alpha-1)}$ (α: $\gamma^1_{\mathbb{C}}$)

② Discuss $K^0 B\mathbb{Z}/k$.

Complex cobordism and the universal FGL

$$\Omega_n^{\mathbb{C}} = \pi_n MU$$

complex cobordism groups

Thom spectrum MU

Theorem of Thom

$$MU = \operatorname{colim}_{k\to\infty} \Sigma^{-2k} BU(k)^{\gamma^k_{\mathbb{C}}}$$

classifying space of complex k-bundles

Thom space with respect to the universal complex k-bundle

$$\Sigma^2 BU(k)^{\gamma^k_{\mathbb{C}}} = BU(k)^{\gamma^k_{\mathbb{C}} \oplus 1_{\mathbb{C}}} \to BU(k+1)^{\gamma^{k+1}_{\mathbb{C}}}$$

$$\begin{array}{ccc} & BU(1) & \\ & \sim & \\ \Sigma^{-2}\mathbb{C}P^\infty & \longrightarrow & MU \\ \uparrow & \nearrow & \\ \Sigma^{-2}\mathbb{C}P^1 \simeq S & & \end{array}$$

This is equivalent to complex orientation

Theorem (Quillen, Novikov): The FGL associated with MU is the universal formal group law (In particular, $\pi_* MU = L$)

Lazard ring.

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Familiarising ourselves with the universal FGL.

(Appendix 2 of Ravenel: Complex cobordism and stable homotopy groups of spheres)

Over a commutative ring R, a FGL is a formal series

$$\underset{x+_F y}{\overset{}{F(x,y)}} \in R[[x,y]]$$

$F(x,0) = F(0,x) = x$

$F(x,y) = F(y,x)$

$F(x, F(y,z)) = F(F(x,y), z)$.

Universal FGL:

$$F(x,y) = \sum_{k,l \geq 0} a_{kl} x^k y^l$$

$$L = \mathbb{Z}[a_{k,l} \mid k,l \geq 0] \Big/ \text{these axioms on coeffs.}$$

$\ldots$

$a_{kl} = a_{lk}$

$\ldots$

Lazard ring

Exercise: write down the other relations

$$\boxed{\{\text{FGL on } R\} \cong \mathrm{Mor}_{\text{rings}}(L,R)}$$

$\underbrace{\text{Strict isomorphism}}_{\text{reparametrization}}$ of Formal group laws on same ring R $\quad f : F \to G$

$$f(x) = x + b_1 x^2 + b_2 x^3 + \cdots \quad \in R[[x]]$$

strict : $1 \cdot x$

$$G(f(x), f(y)) = f(F(x,y))$$

Up to strict isomorphism, the FGL of a cx. oriented generalized cohomology does not depend on the choice of complex orientation.

Theorem : For any commutative ring R, an FGL F on $R \otimes \mathbb{Q}$ is strictly isomorphic to $+$ $(x+y)$.

The strict isomorphism $F \to +$ is called the formal logarithm.

$$\log_F(x) = x + m_1 x^2 + m_2 x^3 + \cdots$$