Th: If R is a commutative ring, F is a FGL on R, then in $R\otimes\mathbb{Q}$, there exists a strict isomorphism $\log_F(x) : F \to +$.

$$\log_F(F(x,y)) = \log_F x + \log_F y \qquad (*)$$

$$F(x,y) = \sum_{i,j\geq 0} a_{ij} x^i y^j \in R[[x,y]]$$

$$\log_F x = x + m_1 x^2 + m_2 x^3 + \cdots \in R\otimes\mathbb{Q}[x]$$

Proof: Explicit formula:

$$f(x) := \log_F(x) = \int_0^x \frac{dt}{\frac{\partial}{\partial y} F(t,y)\big|_{y=0}}$$

We need to verify $(*)$.

$$w(x,y) = f(F(x,y)) - f(x) - f(y)$$

$$\frac{\partial w}{\partial y} = f'(F(x,y))\frac{\partial}{\partial y}F(x,y) - f'(y) = \underbrace{\frac{1}{\frac{\partial}{\partial z}F(F(x,y),z)\big|_{z=0}} \cdot \frac{\partial}{\partial y}F(x,y) - \frac{1}{\frac{\partial}{\partial z}F(y,z)\big|_{z=0}}}_{\circledast}$$

need to show $\circledast = 0$

$③ = ① \cdot ②$

$$\underbrace{\frac{\partial}{\partial z} F(F(x,y),z)}_{③}\Big|_{z=0} = \frac{\partial}{\partial z} F(x, F(y,z))\Big|_{z=0} =$$

$$= \underbrace{\frac{\partial}{\partial t} F(x,t)\Big|_{t=F\underbrace{(y,0)}_{y}}}_{\frac{\partial}{\partial y} F(x,y) \; ①} \cdot \underbrace{\frac{\partial}{\partial z} F(y,z)}_{②}\Big|_{z=0}$$

$$\frac{1}{③} ① = \frac{1}{②}$$

$$⊛: \quad \frac{1}{③ \; \frac{\partial}{\partial z} F(F(x,y),z)\Big|_{z=0}} \cdot \underbrace{\frac{\partial}{\partial y} F(x,y)}_{①} = \frac{1}{\underbrace{\frac{\partial}{\partial z} F(y,z)}_{②}\Big|_{z=0}}$$

②

$$\Big| \cdot \frac{\partial}{\partial z} F(y,z)\Big|_{z=0} \cdot \underbrace{\frac{\partial}{\partial z} F(F(x,y),z)}_{③}\Big|_{z=0}$$

$\square$

How to compute with FGL?

$$\log_F(x) = x + m_1 x^2 + m_2 x^3 + \cdots$$

A formal power series $\in R[[x]]$ which starts with x has a formal inverse $\in R[[y]]$

$$\exp_F(y) = y + b_1 y^2 + b_2 y^3 + \cdots$$

$$x = \exp_F(\log_F(x)) = \underbrace{\log_F x}_{x + m_1 x^2 + \cdots} + b_1(\underbrace{\log_F x}_{x + m_1 x^2 + \cdots})^2 + \overbrace{b_2(\underbrace{\log_F x}_{x + m_1 x^2 + \cdots})^3 + \cdots}^{\geq \text{cubic in } x}$$

This is a recursive formula for the b_n's in terms of the m_n's.

$b_1 x^2 + b_1(\geq$ cubic in $x)$

$b_2 x^2$ + HOT in x

Calculating explicitly is of interest in analytic function thy

(? inverse of an analytic function analytic?)

<u>Lagrange</u>

Lagrange inversion formula

$$f(x) = x + \text{HOT in } x$$

$$g(y) = f^{-1}(y) = \sum_{n=1}^{\infty} g_n y^n$$

$$g_n = \frac{1}{n!} \frac{\partial^{n-1}}{\partial w^{n-1}} \left(\left(\frac{w}{f(w)} \right)^n \right)_{w=0}$$

A formal group law can be calculated as

$$F(x,y) = \exp_F (x + m_1 x^2 + m_2 x^3 + \cdots + y + m_1 y^2 + m_2 y^3 + \cdots)$$

use lagrange formula: coefficients of $F(x,y)$ are expressed in terms of the m_i's.

$$\exp_F(y) = y + b_1 y^2 + b_2 y^3 + \cdots$$

$$b_{n-1} = \frac{1}{n!}\left(\frac{\partial^{n-1}}{\partial w}\left(\frac{w}{\underbrace{w_1 + m_1 w^2 + \cdots}}\right)^n\right)\Bigg|_{w=0}$$

$$\left(\frac{1}{1 + m_1 w + m_2 w^2 + \cdots}\right)^n$$

Example:

$$b_1 = \frac{1}{2}\left(\frac{\partial}{\partial w}\left(\frac{1}{1 + m_1 w + \cdots}\right)^2\right)\Bigg|_{w=0}$$

$n=2$

$$= \frac{1}{2}(-2m_1) = -m_1$$

$$F(x,y) = x + y + \underbrace{m_1 x^2 + m_1 y^2 - m_1(x+y)^2}_{-2m_1 xy} + HOT$$

In the ring $\mathbb{Z}[m_1, m_2, \ldots] \Big/ \begin{pmatrix} m_i, & i \neq n \\ m_i^2, & i \in \mathbb{N} \end{pmatrix}$

Modulo

$$F(x,y) = x + y + m_n x^{n+1} + m_n y^{n+1} - m_n (x+y)^{n+1}$$

I recover a multiple of m_n which is: $\gcd\left\{\binom{n+1}{k} \,\middle|\, 1 \le k \le n\right\}$

$= 1$ if $n+1$ is not a prime power

p if $n+1 = p^j$.

$\binom{3}{1}, \binom{3}{2}$ } $\gcd = 3$

$\binom{4}{1}, \binom{4}{2}, \binom{4}{3}$ } $\gcd = 2$

$\binom{6}{1}^6, \binom{6}{2}^{15}, \binom{6}{3}^{20}, \binom{6}{4}^{15}, \binom{6}{5}^6$ } $\gcd = 1$

$F(x,y) = \sum a_{ij} x^i y^j$

Lazard's theorem: The Lazard ring $L = \mathbb{Z}[a_{ij}]/\text{FGL relations}$

coefficients of $\log_F x = x + m_1 x^2 + \dots$ or the coefficients a_{ij}

is a subring of $\mathbb{Z}[m_1, m_2, \dots]$ of the form $\mathbb{Z}[x_1, x_2, \dots]$ where modulo $\underbrace{\text{monomials in } m_i \text{ of total degree} \geq 2}_{\text{decomposables}}$,

$$x_i \equiv \begin{cases} m_i & \text{if } i+1 \text{ is not a prime power} \\ p m_i & \text{if } i+1 = p^k. \end{cases} \qquad \square$$

What we said using the Lagrange formula proves this modulo the fact that L has no torsion.

Appendix 2 to Ravenel: Complex cobordism and stable homotopy groups of spheres.