Pontrjagin classes and the Euler class

The Euler class is "the top Chern class" for an oriented bundle.
$E = H\mathbb{Z}$, ξ be an oriented real n-bundle on X. So there exist a Thom class $u \in \tilde{H}^n(X^\xi;\mathbb{Z})$

[$\Theta^*: H^k(X;\mathbb{Z}) \otimes \tilde{H}^n(X^\xi;\mathbb{Z}) \to \tilde{H}^{n+k}(X^\xi;\mathbb{Z})$

$\Theta^*(? \otimes u): H^k(X;\mathbb{Z}) \xrightarrow{\cong} \tilde{H}^{n+k}(X^\xi;\mathbb{Z})$ ← Thom isomorphism

We have a 0-section map

$$X_+ \xrightarrow{\sigma} X^\xi$$

$$\boxed{e(\xi) := \sigma^*(u) \in H^n(X;\mathbb{Z})}$$ ← Euler class.

<u>Claim</u>: If n is odd then $2e(\xi) = 0$

Proof: If we $\underbrace{\text{reverse the orientation}}_{u \leftrightarrow -u}$ then the Euler class changes sign.

We can also think of $x \to -x$ on the total space of ξ and also on X^ξ.
If n is odd, this will reverse orientation, but it will preserve the 0-section:

$$\begin{tikzcd}
 & X^\xi \arrow[d, "x \mapsto -x"] \\
X_+ \arrow[ur, "\sigma"] \arrow[r, "\sigma"'] & X^\xi
\end{tikzcd}$$

$\therefore\ e(\xi) = -e(\xi)$. $\square$

Pontryagin classes: Let ξ be an n-bundle on X, not necessarily oriented. Consider the complexified bundle (a complex n-bundle)

$$\xi_{\mathbb{C}} = \xi \otimes_{\mathbb{R}} \mathbb{C}$$

We can consider the k-th Chern class:

$$c_k(\xi_{\mathbb{C}}).$$

<u>Claim:</u> If k is odd, then $2c_k(\xi_{\mathbb{C}}) = 0$.

Proof: $\bar{\xi}_{\mathbb{C}} \cong \xi_{\mathbb{C}}$. What does complex conjugation do to Chern classes?

complex conjugate.

$$H^* BU(n) = (H^* \underbrace{\overset{BU(1)}{\mathbb{C}P^\infty} \times \cdots \times \mathbb{C}P^\infty}_{n})^{\Sigma_n} = \mathbb{Z}[\underbrace{x_1, \dots x_n}]^{\Sigma_n}$$

On $\mathbb{C}P^\infty$, $\gamma^1_{\mathbb{C}} \leftrightarrow \bar{\gamma}^1_{\mathbb{C}}$, $x_1 = e(\gamma^1_{\mathbb{C}})$, $x_1 \leftrightarrow -x_1$

reverses orientation

complex conjugation changes the signs of x_i, therefore also of $\sigma_k(x_1, \dots x_n)$, k odd. $\therefore$ of c_k. $\square$

Our focus is on coefficient $\mathbb{Z}[\frac{1}{2}]$, we put ξ is a real n-bundle

$\xi_{\mathbb{C}}$ is a cx. n-bundle

$$\boxed{p_k(\xi) := c_{2k}(\xi_{\mathbb{C}})} \in H^{4k}(X;\mathbb{Z}),$$

this makes sense for $2k \le n \iff k = 1, \dots \lfloor \frac{n}{2} \rfloor$.

Pontryagin class

Theorem: $H^*(BSO(n); \mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}][p_1, \dots, p_{\lfloor \frac{n}{2} \rfloor}]$ for n odd

polynomial ring

$= \mathbb{Z}[\frac{1}{2}][p_1, \dots, p_{\frac{n}{2}-1}, e]$

$e^2 = p_{\frac{n}{2}}$

oriented real n-dimensional vector subspace of $\mathbb{R}^{\oplus\infty}$

$[X, BSO(n)] = \{\cong$ classes of oriented real n-bundles on $X\}$

if X paracompact.

$H^*(BO(n); \mathbb{Z}[\frac{1}{2}])$ is the same with the Euler class omitted.

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Proof less pretty than for Chern classes — no Schubert calculus

Other methods.

Gysin sequence: If X is a CW-complex, ξ is a real vector n-bundle on X, then we have a Euclidean metric on ξ, so we can talk about the unit disk bundle $D(\xi)$ and the unit sphere bundle $S(\xi) \overset{i}{\subseteq} D(\xi)$.

cofibration

$\underbrace{D(\xi)/S(\xi)}_{\simeq\, C_i} \cong X^{\xi}$. The Gysin sequence:

$$S(\xi)_+ \longrightarrow X_+ \longrightarrow X^{\xi}$$

based spaces

"Thomifying" the Gysin sequence: If η is a vector bundle (or a virtual bundle), on X

pullback of η to $S(\xi)$

$$S(\xi)^{\eta} \longrightarrow X^{\eta} \longrightarrow X^{\xi \oplus \eta}$$

based spaces when η is a bundle

spectra when η virtual

In the case of the universal real n-bundle $\gamma^n = \gamma^n_{\mathbb{R}}$ on $BO(n)$, we get the Gysin sequence:

$$BO(n-1)_+ \longrightarrow BO(n)_+ \longrightarrow BO(n)^{\gamma^n}$$

$$\boxed{BSO(n-1)_+ \longrightarrow BSO(n)_+ \longrightarrow BSO(n)^{\gamma^n}}$$

$(H\mathbb{Z})$

γ^n is oriented

$\Downarrow$

coeffs. $\mathbb{Z}$, $\mathbb{Z}[1/2]$

$$H^{k+1-n}(BSO(n)) \longleftarrow H^k(BSO(n-1)) \longleftarrow H^k(BSO(n)) \longleftarrow H^{k-n}(BSO(n)) \longleftarrow$$

Note that the answer I proposed would satisfy this:

n even

by definition of e

$$0 \leftarrow \mathbb{Z}[1/2][p_1,\dots,p_{\frac{n}{2}-1}] \leftarrow \mathbb{Z}[1/2][p_1,\dots,p_{\frac{n}{2}-1},e] \xleftarrow{\;e\;} \mathbb{Z}[1/2][p_1,\dots,p_{\frac{n}{2}-1},e] \leftarrow 0$$

n odd

$$0 \leftarrow \mathbb{Z}[1/2][p_1,\dots,p_{\lfloor\frac{n}{2}\rfloor}]e \leftarrow \underbrace{\mathbb{Z}[1/2][p_1,\dots,p_{\lfloor\frac{n}{2}\rfloor-1},e]}_{\mathbb{Z}[1/2][p_1,\dots,p_{\lfloor\frac{n}{2}\rfloor}] \,\oplus\, e\,\mathbb{Z}[1/2][p_1,\dots,p_{\lfloor\frac{n}{2}\rfloor}]} \leftarrow \mathbb{Z}[1/2][p_1,\dots,p_{\lfloor\frac{n}{2}\rfloor}] \leftarrow 0$$

$e^2 = p_{\lfloor\frac{n}{2}\rfloor}$