Math 696

Friday, January 31, 2025 8:57 AM

Cofibration:

$$BSO(n-1)_+ \to BSO(n)_+ \to BSO(n)^{\gamma^n}$$ ← universal oriented real n-bundle

Gysin exact sequence:

$$\cdots \leftarrow H^q BSO(n-1) \leftarrow H^q BSO(n) \leftarrow H^{q-n} BSO(n) \leftarrow H^{q-1} BSO(n-1) \leftarrow \cdots$$

coeffs. $\mathbb{Z}[\frac{1}{2}]$

We can turn this into the Gysin spectral sequence

$$\cdots \leftarrow E \leftarrow D \leftarrow D \leftarrow E \leftarrow \cdots$$

cohomological

$$D_1^{pq} = H^{q+(1-n)p} BSO(n)$$

$$E_1^{pq} = H^{q+(1-n)p} BSO(n-1)$$

$$E_1 = H^* BO(n-1)[u] \Rightarrow H^* BO(n)$$

$$|u| = (1, n-1)$$

$(n-1)$ odd then all total degrees even $\Rightarrow$ Spectral sequence collapses

$u \rightsquigarrow e$

$(n-1)$ even then $d_1(e) = u$. Spectral sequence of rings.

$$H^* BSO(n-1)_+ \leftarrow H^* BSO(n)_+$$

d': $H^* BSO(n-1)^{\gamma^n} \leftarrow \gamma^n \oplus 1$ ⫴ Thom iso ⫴ Thom iso

$$H^* \Sigma BSO(n-1)_+ \to H^* BSO(n)^{\gamma^n}$$ } "degree 1 on the fiber"

Milnor - Stasheff convention: just a sign

$$\boxed{p_n(\xi) := (-1)^n c_{2n}(\xi \otimes_{\mathbb{R}} \mathbb{C})}$$

$$H^*(BSO(n), \mathbb{Z}[\tfrac{1}{2}]) = \mathbb{Z}[\tfrac{1}{2}][p_1, \dots, p_{\lfloor \frac{n}{2} \rfloor}] \quad n \text{ odd}$$

$$\mathbb{Z}[\tfrac{1}{2}][p_1, \dots p_{\frac{n}{2}-1}, e] \quad n \text{ even}$$

$$(e^2 = (\pm)\, p_{\frac{n}{2}})$$

Basic example of a characteristic number:

If M is an oriented compact n-manifold, then

$e[M] = \langle e(T_M), [M] \rangle = \chi(M)$ ← vector field v

Geometric argument: take a $\underbrace{\text{section}}$ of T_M ⋔ 0-section

\# of intersections with the 0-section

(with appropriate signs given by orientation)

= \# 0's of the vector field v (with orientation signs) =

(Lefschetz theorem) $\lambda(Id_M) = \chi(M)$.

With Pontryagin numbers we can do more (Chapter 19 of Milnor-Stasheff)

A compact oriented $4k$-manifold M has an invariant called the signature $\sigma(M)$. Poincaré duality ($\otimes \mathbb{Q}$)

$$H^i(M;\mathbb{Q}) \otimes H^{4k-i}(M;\mathbb{Q}) \to H^{4k}(M;\mathbb{Q}) = \mathbb{Q}$$

finite dim. v. spaces

$i = 2k$

quadratic form on

$$H^{2k}(M;\mathbb{Q}) \text{ ; in particular on } H^{2k}(M;\mathbb{R}).$$

A quadratic form over $\mathbb{R}$ has a signature (Sylvester's Thm: diagonalise the matrix of the form, $\#>0$ eigenvalues $- \# <0$ eigenvalues)

$\Rightarrow$ invariant : $\sigma(M)$ = signature

Signature: A homomorphism of rings

← oriented cobordism ring

$$\Omega^{\text{oriented}} \longrightarrow \mathbb{Z}$$

0 in dimensions not divisible by 4

Remark: Stiefel-Whitney and Pontryagin numbers determine the class of a compact oriented manifold in Ω^{oriented}.

Only non-torsion in dim. divisible by 4.

We will be able soon to calculate $\Omega^{\text{oriented}} \otimes \mathbb{Q}$

$$= \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \dots]$$

This implies that $\sigma(M)$ has to be calculated in terms of Pontryagin numbers.

What types of sequences of Pontryagin numbers give homomorphisms of rings $\Omega^{\text{oriented}} \to \mathbb{Q}$?

Such a sequence is called a genus.

Each genus is given by a generating series

$$K(x) = 1 + K_1 x + K_2 x^2 + \cdots$$

$$K(x_1)\cdots K(x_n)\cdots = 1 + K_1(\sigma_1(x_1, x_2, \cdots)) + K_2(\sigma_1, \sigma_2) + K_3(\sigma_1, \sigma_2, \sigma_3) + \cdots$$

K-genus of an oriented 4n-manifold M

elementary symmetric polynomials

$$K[M] = \langle K_n(p_1(T_M), \cdots p_n(T_M)), [M] \rangle$$

The generating series for $\sigma[M]$ is

Bernoulli number

$$L(z) = \frac{\sqrt{z}}{\tanh\sqrt{z}} = 1 + \frac{1}{3}z - \frac{1}{45}z^2 + \cdots + (-1)^{k-1}\frac{2^{2k} B_k z^k}{(2k)!} + \cdots$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Theorem: $\sigma(M) = L[M]$.

$$\sigma(\mathbb{C}P^{2k}) = 1 \qquad H^*(\mathbb{C}P^{2k};\mathbb{Z}) = \mathbb{Z}[a]/a^{2k+1}$$

Total Pontryagin class $p(\mathbb{C}P^{2k}) = (1+a^2)^{2k+1}$ $\quad |a|=2$

$$L(1+a^2+0+0\cdots) = \frac{\sqrt{a^2}}{\tanh\sqrt{a^2}} = \frac{a}{\tanh a}$$

? coeff at z^{2k} of $\left(\frac{z}{\tanh z}\right)^{2k+1}$.

coeff @ u^{-1}

$$\frac{1}{2\pi i}\oint\frac{dz}{(\tanh z)^{2k+1}} = \frac{1}{2\pi i}\oint\frac{du\,(1+u^2+u^4+\cdots)}{u^{2k+1}} = 1$$

$z = \operatorname{arctanh} u \qquad dz = \frac{du}{1-u^2} = (1+u^2+\cdots)\,du$

$\square$

Alternative proof: Atiyah-Singer index theorem.
another application
(Riemann-Roch theorem for compact Kähler manifolds)