Math 696

Monday, February 3, 2025 8:55 AM

Milnor's famous work: Defined Pontryagin numbers for PL-manifolds (triangulated), 8-manifold with non-integral Pontryagin numbers

∴ no smooth structure. } Chapter 20 of Milnor-Stasheff

gluing two smooth compact 7-manifolds with boundary where the boundary is homeomorphic to a 7-sphere ∴ ∃ non-diffeomorphic smooth structures on 7-spheres. Kervaire-Milnor: classification of smooth structures on spheres.

How do we really compute cobordism groups?

Think about examples:

- unoriented
- oriented
- complex
- framed: Compact smooth manifold M, $\tau_M \oplus \nu \underset{\sim}{\xrightarrow{\Xi}} N$ (trivial bundle)

$\nu \xrightarrow{\Xi} k$

$\dim M = N - k$

Recall that cobordism groups are isomorphic to the homotopy groups (coefficients) of the corresponding Thom spectrum.

The most important example for developing homotopy theory is complex cobordism. Thom spectrum: MU.

$$MU = \operatorname{colim} \Sigma^{-2k} BU(k)^{\gamma^k_{\mathbb{C}}}$$

$$\Sigma^2 BU(k)^{\gamma^k_{\mathbb{C}}} = BU(k)^{\gamma^k_{\mathbb{C}} \oplus 1_{\mathbb{C}}}$$

$$\downarrow$$

$$BU(k+1)^{\gamma^{k+1}_{\mathbb{C}}}$$

Thomification of classification of $\gamma^k_{\mathbb{C}} \oplus 1_{\mathbb{C}}$

how we compute homotopy groups

$$\pi_n MU = \pi_{n+2k} BU(k)^{\gamma^k_{\mathbb{C}}} \qquad k >> 0 \ (\sim n)$$

Let

M be a smooth compact weakly stable complex n-manifold.

$$\tau_M \oplus \nu = N \quad \text{"} n+2k$$

ν is given a structure of a complex k-bundle.

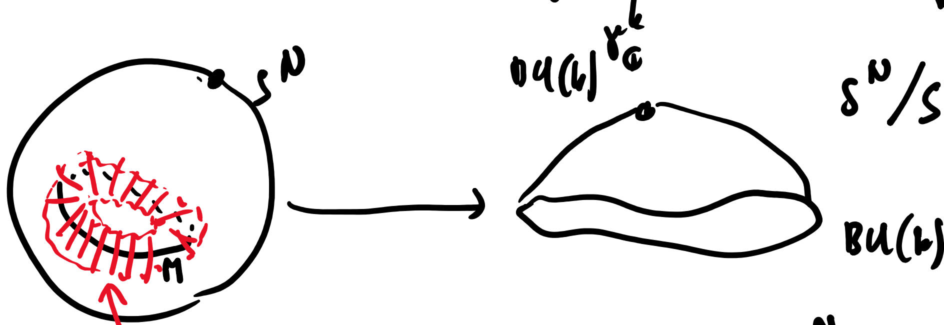
$$M \subset \mathbb{R}^N \qquad \nu = \nu_M^{\mathbb{R}^N}.$$

View S^N as a 1-point compactification of $\mathbb{R}^N$. $N = n + 2k$

$M \subset S^N \setminus \{*\}$

$M \xrightarrow{\varphi} BU(k)$ ← classifies ν

$S^N / S^N \setminus U = M^\nu$



U tubular neighborhood
$\cong$ total space of ν ← complex k-bundle
$\dim_{\mathbb{R}} \nu = 2k$

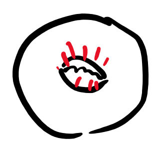
Thom construction

$$\underbrace{\underbrace{S^N \to S^N / S^N \setminus U = M^\nu \xrightarrow{\varphi^{\gamma^k_{\mathbb{C}}}} BU(k)^{\gamma^k_{\mathbb{C}}}}_{\in \pi_N BU(k)^{\gamma^k_{\mathbb{C}}}}}_{\text{Pontryagin-Thom construction}}$$

Why complex cobordant manifolds give the same homotopy group?

Doing the same on the cobordism N gives the homotopy.

$\partial N = M_1 \sqcup M_2$



How do we go back? $N = n + 2k$

$$S^N \xrightarrow{\alpha} BU(k)^{\gamma^k_{\mathbb{C}}}$$

0-section

inclusion

$BU(k)$

It makes sense to say $\alpha \pitchfork$ to the 0-section. } Lemma: this can always be arranged.

transverse

Milnor-Stasheff (Chapter 16 or 17)

Then $M = \alpha^{-1}(0\text{-section}_{BU(k)})$ is a stably weakly complex n-manifold.

Cobordism: doing the same thing on homotopy

$$[0,1]_+ \wedge S^N \xrightarrow{h} BU(k)^{\gamma^k_{\mathbb{C}}}$$

$\underbrace{[0,1]}_{\text{manifold with } \partial} \times S^N / [0,1] \times *$

h being $\pitchfork$ 0-section.

recall the Hopf theorem: $\underbrace{\pi_n S^n = \mathbb{Z}}_{\text{cobordism of framed 0-mflds.}}$

Milnor: Topology from a differential viewpoint.

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We proved Thom's theorem:

$$\Omega_n^{\text{complex}} = \pi_n MU$$

Variants:

$$\Omega_n^{\text{unoriented}} = \pi_n MO \qquad MO = \operatorname{colim}_{\to} \Sigma^{-k} BO(k)^{\gamma_{\mathbb{R}}^k}$$

L-genus $\rightsquigarrow$ $$\Omega_n^{\text{oriented}} = \pi_n MSO \qquad MSO = \operatorname{colim}_{\to} \Sigma^{-k} BSO(k)^{\gamma_{\mathbb{R}}^k}$$

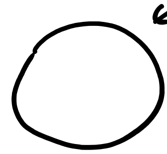
$$\Omega_n^{\text{framed}} = \pi_n S$$

sphere spectrum

(Hopf's theorem is a special case of Thom's Theorem for framed cobordism, $n=0$.)

We can certainly use framed cobordism to compute $\pi_1 S$

← 2 copies of Möbius band $\cong$ trivial



Pontryagin incorrectly computed $\pi_2 S = 0$

In fact, it is also $\mathbb{Z}/2$

Framing periodic in all directions is non-trivial (PP spin structure)

The progress in discussion of homotopy theory: discussing the Thom spectra. MO, MSO, MU.

Let's begin with MSO, finishing the proof of the signature thm.

Recall that if E is a spectrum, $E_{\mathbb{Q}} = \operatorname{colim} E \to E \to E \to E \to \cdots$

all positive integers

$\pi_n E_{\mathbb{Q}} = (\pi_n E) \otimes \mathbb{Q}$

(replace E by an abelian group A, this is how you construct $A \otimes \mathbb{Q}$).

$E_{\mathbb{Q}} = E \wedge S_{\mathbb{Q}}$

$S_{\mathbb{Q}} = H\mathbb{Q}$ $\quad \left(\begin{array}{l} \pi_0 S = \mathbb{Z} \\ \pi_n S = \text{finite} \quad n > 0 \end{array} \right)$

$$(\pi_n MSO) \otimes \mathbb{Q} = \pi_n MSO_{\mathbb{Q}} \simeq H_n(MSO;\mathbb{Q})$$

$\cong$ Thom isomorphism theorem

$H_n(BSO;\mathbb{Q})$

$\|$

dual of $H^n(BSO;\mathbb{Q})$

$H^*(BSO;\mathbb{Q}) = \mathbb{Q}[p_1, p_2, \cdots]$

We have proved that Pontryagin numbers are a complete system of invariants for $\Omega^{\text{oriented}} \otimes \mathbb{Q}$.

For the signature theorem, we specifically said it sufficed to prove it for $\mathbb{C}P^{2k}$.