? Find multiplicative generators of $\underbrace{H_*(MSO;\mathbb{Q})}_{\cong} = \Omega^{\text{oriented}} \otimes \mathbb{Q}$

commutative $\mathbb{Q}$-algebra

ring by product of manifolds

(We said an answer: $\mathbb{C}P^{2k}$, $k=1,2,\dots$)

$H_*(BSO;\mathbb{Q})$ ← ring by $\otimes$ real oriented vector bundles

$H^*(BSO;\mathbb{Q}) = \mathbb{Q}[p_1,p_2,p_3,\dots]$

dual Hopf algebras

<u>Hopf algebra</u> ← product, coproduct ψ
unital co unital

$\psi(ab) = \psi(a)\psi(b)$

graded

$|p_i| = 4i$

$\psi: A \to A \otimes A$

Whitney formula on $H^*(BSO;\mathbb{Q})$: $\psi(p_n) = \sum_{k=0}^{n} p_k \otimes p_{n-k}$ $(p_0 = 1)$

This Hopf algebra is bipolynomial. Both A and the dual $A^\vee$ are polynomial $\mathbb{Q}$-algebras. ? Find polynomial generators of $H_*(BSO;\mathbb{Q})$.

<u>Dual notion</u> to graded the coproduct of a Hopf alg. B:

<u>primitives</u>:

$\psi(x) = 1 \otimes x + x \otimes 1$

↖ homogeneous

$PB \xrightarrow{\varepsilon} B$

$P(A^\vee) = (QA)^\vee$

Suppose I have a graded polynomial algebra

$A = \mathbb{Q}[a_1, a_2, \dots]$

How do I find a general set of polynomial generators of A?

a_n + decomposable element (of polynomial order >1)

$A \twoheadrightarrow QA = \mathbb{Q}\{a_1, a_2, \dots\}$ ← basis

↖ indecomposables

? Primitives in $H^*(BSO;\mathbb{Q}) \underset{\sim}{=} \mathbb{Q}[p_1, p_2, \cdots]$

$$\psi(p_n) = \sum_{k=0} p_k \otimes p_{n-k}$$

Recipe: Pretend that p_n are elementary symmetric polynomials of some variables $x_1, x_2, \cdots$ $p_n = \sigma_n(x_1, x_2, \cdots)$. $|x_i| = 4i$

A primitive element of degree $4n$:

$$x_1^n + x_2^n + x_3^n + \cdots = s_n(x_1, x_2, \cdots)$$

(The Milnor class). Recursive formula:

The Newton formula

$$s_n - \sigma_1 s_{n-1} + \sigma_2 s_{n-2} - \cdots + (-1)^n \sigma_n n = 0.$$

$(x - r_1) \cdots (x - r_n) = x^n - \sigma_1 x^{n-1} + \cdots + (-1)^n \sigma_n$ $\sigma_k = \sigma_k(r_1, \cdots r_n)$

Plug in $x = r_k$ and sum.

Corollary: Manifolds M_n of dim. $4n$ form a system of polynomial generators of $\Omega^{\text{oriented}} \otimes \mathbb{Q}$ if and only if $s_n[M_n] \neq 0$.

$\dim M_n = 4n$

Example: $M_n = \mathbb{C}P^{2n}$

$H^*(\mathbb{C}P^{2n};\mathbb{Q}) =$
$= \mathbb{Q}[a]/a^{2n+1}$
$|a|$

$p(\gamma_{\mathbb{C}}^1) = 1 + a^2$ $\quad 1 + \tau_{\mathbb{C}P^{2n}} \cong (2n+1)\gamma_{\mathbb{C}}^1$

$s_n (2n+1)\gamma_{\mathbb{C}}^1 = (2n+1) s_n \gamma_{\mathbb{C}}^1$

$s_n[\mathbb{C}P^{2n}] = \pm(2n+1) \neq 0.$

Can we apply the same technique to $\underbrace{\text{integrality questions?}}_{\Omega^{\text{unoriented}},\ \Omega^{\text{cx.}}}$

$\Omega^{\text{unoriented}}$ ← Stiefel-Whitney numbers; $\Omega^{\text{cx.}}$ ← Chern numbers

We have Milnor Stiefel-Whitney and Chern classes defined by the same formula (the Newton formula).

Can we find, say, unoriented compact smooth manifolds M_n of dim n with

$s_n[M^n] = 1$

← Stiefel-Whitney Milnor numbers

$\mathbb{RP}^n$ works for n even
we are supposed to also have odd ones for $n \neq 2^k - 1$.

<u>Milnor manifolds</u>:

$\mathbb{RP}^k \times \mathbb{RP}^\ell$

$\gamma^1_{\mathbb{R}} \boxtimes \gamma^1_{\mathbb{R}} = \gamma^1_{\mathbb{RP}^k} \otimes \gamma^1_{\mathbb{RP}^\ell}$ ← addition on w_1

section s ↗ to the 0-section

$M :=$ 0 locus of s — smooth manifold of dim. $k+\ell-1$

Calculating the Milnor number: $k,\ell > 0$

$$H^*(\mathbb{RP}^k \times \mathbb{RP}^\ell; \mathbb{Z}/2) = \mathbb{Z}/2[a,b]/(a^{k+1}, b^{\ell+1})$$

with a, b coming from $\mathbb{RP}^k$, $\mathbb{RP}^\ell$; $|a| = |b| = 1$

$? \ s_{k+\ell-1}[M]$

normal bundle of M in $\mathbb{RP}^k \times \mathbb{RP}^\ell$: ν

$\tau_M = \tau_{\mathbb{RP}^k} + \tau_{\mathbb{RP}^\ell} - \nu$

$w_1(\nu) = a+b$ $\qquad$ $w_j(\nu) = (a+b)^j$ $\qquad$ $s_j(\tau_M) =$

$j = k+\ell-1$, pair with $[M]$ $\longrightarrow$ $(k+1)a^j + (\ell+1)b^j - (a+b)^j$

$j = k+\ell$, pair with $[\mathbb{RP}^k \times \mathbb{RP}^\ell]$

$= \operatorname{coeff}_{a^k b^\ell}\left(\cancel{(k+1)}a^{k+\ell} + \cancel{(\ell+1)}b^{k+\ell} - (a+b)^{k+\ell}\right) = \binom{k+\ell}{k}$

We proved

Proposition: $s_{k+\ell-1}[M] = \binom{k+\ell}{k}$. $(k,\ell>0)$.

Milnor manifold, = degree (1,1) hypersurface in $\mathbb{RP}^k \times \mathbb{RP}^\ell$

odd when k,ℓ share no 1 in any 2-expansion place. This exist for any $k+\ell \neq 2^m$.

The Milnor manifold form polynomial generators of $\Omega^{\text{unoriented}}$.

Cohomological operations (Steenrod operations) $\Big\{$ Why aren't there any (2^m-1)-manifolds M with $s_{2^m-1}[M]$ odd?

We can also play the same game for complex cobordism $\Omega^{\mathbb{C}}$. $s_k = s_k^{\text{chern}}$.

Two changes: ① $k = p^m - 1$ $\quad \binom{p^m}{p^{m-1}} = p\cdot a$, $p \nmid a$

Milnor (p^m-1)-mfld: $s_{p^m-1}[M] = p$

② We have all primes at the same time. For $j \neq p^m - 1$, $s_j[M] = 1$ is possible, linear combinations of Milnor mflds.

Conjecture: M can be smooth proj. varieties$/\mathbb{C}$. Settled by Bryan Johnston in his thesis $\sim$ 2000.
Yes.