Math 696

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Steenrod operations (cohomology operations)

coefficient: $\mathbb{Z}/2$.

"Obstructions to commutativity of cochain multiplication"

Eilenberg-Zilber theorem:

coeffs $\mathbb{Z}/2$

$$C_*(X \times Y) \xrightarrow{\sim} C_*(X) \otimes_{\mathbb{Z}/2} C_*(Y)$$

natural in spaces X, Y

$X = Y$:

$$C_*(X) \xrightarrow{\sim} C_*(X) \otimes C_*(X)$$

natural in X

trivial

$\mathbb{Z}/2$ action interchanging factors

?$\exists$ natural $\mathbb{Z}/2$-equivariant (i.e. preserving action) $\rightsquigarrow$ dualising, hope for commutative DGA structure on $C^*(X)$.

Does not exist! (there is an E_∞-action)

$$X \xrightarrow{\Delta} X \times X$$

$\mathbb{Z}/2$ trivial action (on X); $\mathbb{Z}/2$ (on $X \times X$)

"Free it up" $\quad E\mathbb{Z}/2 = S(\infty) =$ unit sphere in $\mathbb{R}^{\oplus\infty}$

$x \overset{\mathbb{Z}/2}{\longleftrightarrow} -x$

$C_* E\mathbb{Z}/2$ is a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\mathbb{Z}/2]$-module

$$R \quad \cdots \to \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\gamma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+\gamma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\gamma} \mathbb{Z}[\mathbb{Z}/2]$$

γ generator of $\mathbb{Z}/2$

Equivariant form of EZ:

$\mathbb{Z}/2$-fixed

$$\underbrace{C_* E\mathbb{Z}/2}_{R} \otimes C_*(X) \xrightarrow[\substack{\text{natural} \\ \mathbb{Z}/2\text{-equivariant}}]{\sim} C_*(X) \otimes C_*(X) \quad \circlearrowleft \mathbb{Z}/2$$

May: An algebraic approach to Steenrod ops. in LNM 168

Duality:

$$C_* E\mathbb{Z}/2 \otimes C^*(X) \otimes C^*(X) \longrightarrow C^*(X)$$

$$C_* E\mathbb{Z}/2 \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \left(C^*(X) \otimes C^*(X) \right) \longrightarrow C^*(X) \qquad (\dagger)$$

Taking (co)homology (chain). Grade cohomologically. $\mathbb{Z}/2$ acts by a permutation op. on $\underbrace{H^* X}_{\text{basis } (a_i)_{i \in I}} \otimes H^* X$

$|e_k| = -k$ graded cohomologically generator of $H_k(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2$

0 homology

(Co)homology of $(\dagger)$: $\mathbb{Z}/2\{a_i \otimes a_j \mid i \neq j \in I\} \oplus \underbrace{\mathbb{Z}/2\{e_k\}\,\{a_i \otimes a_i \mid i \in I\}}_{\underbrace{|e_k \otimes a_i \otimes a_i| = 2|a_i| - k}_{Sq^j a_i \quad |Sq^j a_i| = i + j}}$

$$|a_i| + j = 2|a_i| - k$$

<u>Basic properties</u>: let X be a space, $a \in H^k X$, $Sq^j a$ is defined for $j \leq k$

(Sometimes we put by convention $Sq^j a = 0$ $j > k$)

(1) $Sq^k a = a^2$ (by definition)

(2) Whitney formula: $Sq^k(ab) = \sum_{i+j=k} Sq^i(a)\, Sq^j(b)$ (also direct)

(3) $Sq^j a = 0$ for $j < 0$, $Sq^0(a) = a$ (hard from this def.)

(4) Compatibility with suspension.

$\therefore$ $Sq^k a$, $a \in H^* X$ X is a suspension (1), (2) do not apply.

<u>Recall from last time</u>. Put $Sq(x) = x + \sum_{j>0} Sq^j x$ $x \in H^* X$ ← space

Whitney: $Sq(xy) = Sq(x)\, Sq(y)$

$X = \mathbb{R}P^\infty$. $H^* \mathbb{R}P^\infty = \mathbb{Z}/2[a]$ $|a| = 1$ By (1), $Sq(a) = a + a^2$

$\therefore$ $Sq(a^n) = (a + a^2)^n$

e.g.

mod 2

$Sq(a^3) = (a + a^2)^3 = a^3 + a^4 + a^5 + a^6$

$Sq^0(a^3) = a^3$
$Sq^1(a^3) = a^4$ $Sq^2(a^3) = a^5$
$Sq^3(a^3) = a^6$

From the point of view of spectra,

$$Sq^k \in H^k(H\mathbb{Z}/2; \mathbb{Z}/2) = H\mathbb{Z}/2^k H\mathbb{Z}/2.$$

EM spectrum

<u>The Steenrod algebra</u> $A^* = H\mathbb{Z}/2^* H\mathbb{Z}/2$ graded (non-commutative $\mathbb{Z}/2$-algebra)

fixed k : A^k has finite rank

We can also consider the dual : $A_* = H\mathbb{Z}/2_* H\mathbb{Z}/2$.

UCT

A_* is a <u>commutative graded algebra</u> because $H\mathbb{Z}/2$ is an ACU ring spectrum. (cup product)

A^*, A_* are dual Hopf algebras. (<u>Exercise</u> - diagram chase involving $\wedge$ of spectra.)

$\mathbb{N}_0$ - graded

connected : $A_0 \cong A^0 = \mathbb{Z}/2$

Milnor : Steenrod algebra and its dual

If X is a space, f.d. CW-cx., then H^*X is an A_*-comodule algebra

$H\mathbb{Z}/2_* H\mathbb{Z}/2$

dualising algebraically the Steenrod operations

$$\lambda : H^*X \longrightarrow H^*X \otimes \underbrace{H\mathbb{Z}/2_* H\mathbb{Z}/2}_{A_*}$$

$$\lambda(ab) = \lambda(a)\lambda(b)$$

infinite sums allowed

(Exercise - extension of spectra diagram chase)

Apply it to $X = \mathbb{R}P^\infty$

$$\lambda : H^*\mathbb{R}P^\infty \longrightarrow H^*\mathbb{R}P^\infty \hat{\otimes} A_*$$

$$\mathbb{Z}/2[a] \xrightarrow{\lambda} \mathbb{Z}/2[a] \hat{\otimes} A_*$$

this is what we are trying to compute

? $\lambda(a)$

$Sq(a) = a + a^2$

$Sq(a^{2^k}) = a^{2^k} + a^{2^{k+1}}$ (mod 2)

$\therefore$ The only non-trivial iterated Steenrod ops. on a are

$$Sq^{2^{k-1}} Sq^{2^{k-2}} \cdots Sq^1 a = a^{2^k}$$

$$\therefore \lambda(a) = \sum_{k=0}^{\infty} a^{2^k} \otimes \xi_k$$

some element of A_{2^k-1}

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$$H^* \mathbb{R}P^\infty = \mathbb{Z}/2[a] \qquad |a| = 1$$

$$\lambda : H^* \mathbb{R}P^\infty \to H^* \mathbb{R}P^\infty \hat{\otimes} A_*$$

$$\lambda(a) = \sum_{k=0}^{\infty} a^{2^k} \otimes \xi_k$$

$\xi_0 = 1$

$k > 0 \quad \xi_k \in A_{2^k - 1}$

Comodule axiom:

$$\underbrace{(\lambda \otimes 1)\lambda(a)}_{\sum a^{2^{k+l}} \otimes \xi_l^{2^k} \otimes \xi_k} = \underbrace{(1 \otimes \psi)\lambda(a)}_{\sum a^{2^m} \otimes \psi(\xi_m)}$$

$\psi : A_* \to A_* \otimes A_*$ is the comultiplication

$$\therefore \quad \boxed{\psi(\xi_m) = \sum_{k=0}^{m} \xi_k^{2^{m-k}} \otimes \xi_{m-k}}$$

Milnor's theorem:

A_* is a commutative graded Hopf algebra

$$\boxed{A_* = \mathbb{Z}/2[\xi_1, \xi_2, \ldots] \quad |\xi_k| = 2^k - 1}$$

We know that rank A_N = rank A^N is not smaller than predicted by this

Why are there all the Steenrod operations? $H\mathbb{Z}/2^* H\mathbb{Z}/2 = \lim \underline{H\mathbb{Z}/2^* K(\mathbb{Z}/2, n)}$