? $H\mathbb{Z}/2^* H\mathbb{Z}/2$

$H\mathbb{Z}/2^* K(\mathbb{Z}/2, n)$

CW-complex $\pi_i K(\mathbb{Z}/2,n) = \begin{cases} \mathbb{Z}/2 & i=n \\ 0 & i\neq n \end{cases}$

$K(\mathbb{Z}/2,1) = \mathbb{RP}^\infty$

$H\mathbb{Z}/2^* \mathbb{RP}^\infty = H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[a]$

$|a| = 1.$

$Sq^1 a = a^2$

$a_1 := a$

$Sq^2 Sq^1 a = Sq^2 a^2 = a^4$

$\vdots$

$\underbrace{Sq^{2^{k-1}} Sq^{2^{k-2}} \cdots Sq^1 a} = a^{2^k}.$

$H\mathbb{Z}/2^* K(\mathbb{Z}/2,2) = \mathbb{Z}/2[a_2, Sq^1 a_2, Sq^2 Sq^1 a_2, \ldots, \underbrace{Sq^{2^{k-1}} Sq^{2^{k-2}} \cdots Sq^1 a_2}_{a_{2,k}} \ldots]$

$|a_2| = 2$

$|a_{2,k}| = 2^k + 1$

$(a_{2,k})^2 = Sq^{2^k+1} a_{2,k} = Sq^{2^k+1} Sq^{2^{k-1}} Sq^{2^{k-2}} \cdots Sq^1 a_2$

multiply by 2, add one

In $H\mathbb{Z}/2^* K(\mathbb{Z}/2,3)$, generator $|a_3| = 3$
a_3

$(a_{2,k})^{2^\ell} = Sq^{2^{\ell-1}(2^k+1)} \cdots Sq^{2^k+1} Sq^{2^{k-1}} \cdots Sq^1 a_2$

all those with a_2 replaced by a_3 are polynomial generators

multiply "exponent" by 2

"gap"

$$H\mathbb{Z}/2^* K(\mathbb{Z}/2, n) = \mathbb{Z}/2[a_n, Sq^{n_1} Sq^{n_2} \cdots Sq^{n_k} a_n \mid n_i \geq 2n_{i+1}, n_k \geq 1, \sum(n_i - 2n_{i+1}) + n_k < k]$$

bound on the "defect" from equalities

Stabilization: $H\mathbb{Z}/2 = \operatorname{colim} \Sigma^{-n} K(\mathbb{Z}/2, n)$ $(= \Sigma^{\infty - n} K(\mathbb{Z}/2, n))$

$$\boxed{H\mathbb{Z}/2^* H\mathbb{Z}/2 = \mathbb{Z}/2\{Sq^{n_1} Sq^{n_2} \cdots Sq^{n_k} \mid n_i \geq 2n_{i+1}, n_k \geq 1\}}$$

free vector space

Induction for $H\mathbb{Z}/2^* K(\mathbb{Z}/2, n)$: Fibration sequence: based

$$F(2) \longrightarrow * \xrightarrow{2} X, \quad F(2) \cong \Omega X$$

Apply to $X = K(\mathbb{Z}/2, n)$:

$$K(\mathbb{Z}/2, n-1) \longrightarrow * \longrightarrow K(\mathbb{Z}/2, n).$$

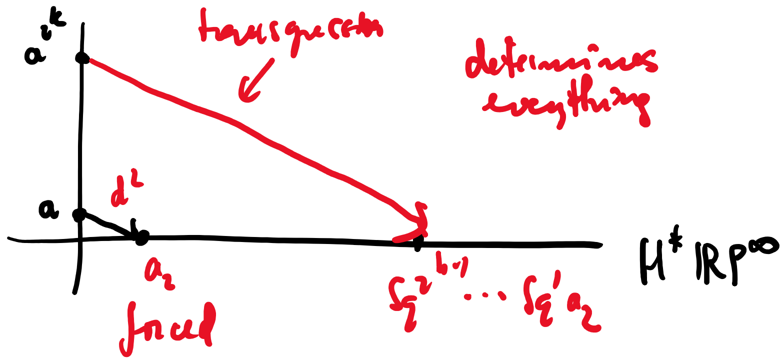
Serre spectral sequence:

(CW-model of base, filter by inverse images total space pr skeleta)

$$\left.\begin{array}{l} H^p(\underbrace{K(\mathbb{Z}/2, n)}; H^q K(\mathbb{Z}/2, n-1)) \Rightarrow \mathbb{Z}/2 \\ n \geq 2 \quad \pi_1 K(\mathbb{Z}/2, n) = 0 \quad \| \\ \underbrace{H^p(K(\mathbb{Z}/2, n); \mathbb{Z}/2)}_{\text{unknown.}} \otimes H^q(K(\mathbb{Z}/2, n-1); \mathbb{Z}/2) \end{array}\right\}$$

$p = q = 0$

Example $n = 2$: $H^*(K(\mathbb{Z}/2, 2))$



$\Rightarrow$ $\mathbb{Z}/2$ in degree $(0,0)$

Spectral sequence of $\mathbb{Z}/2$-algebras
(differentials satisfy the Leibniz rule)

McCleary: User's guide to spectral sequences (corrections on his web page)

Serre's thesis for Serre spectral sequence (popular account in Spanier)

Alternative method: Eilenberg-MacLane spectral sequence

$$\mathrm{Tor}_{H_* \Omega X}(\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow H_* X$$

(have to deal with coalgebras but it collapses).

$$H_*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \bigotimes_{k\geq 1} \Lambda[x_k] \qquad |x_k| = 2^k$$

Hopf algebra dual to

$H^*\mathbb{R}P^\infty = \mathbb{Z}/2[a]$, $\psi(a) = a\otimes 1 + 1\otimes a$

$$\mathrm{Ext}_{\Lambda[x]}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2[y]$$

$|y| = -(|x|+1)$

The EM spectral sequence for $H_*(K(\mathbb{Z}/2,n); \mathbb{Z}/2)$ collapses

$\mathrm{Tor}^{H_*K(\mathbb{Z}/2,n-1)}(\mathbb{Z}/2,\mathbb{Z}/2) \Rightarrow H_* K(\mathbb{Z}/2,n)$.

How does this pair up with Milnor's discussion?

$Sq^{m_1} \cdots Sq^{m_k}$ $\quad m_i \geq 2m_{i+1}$ | vs. monomials in ξ_n

$|\xi_n| = 2^n - 1$

<u>Example</u>:

$$\begin{array}{l|cccccc}
 & \multicolumn{6}{c}{\overbrace{\hphantom{32\ 16\ 8\ 4\ 2\ 1}}^{\xi_6}} \\
 & 32 & 16 & 8 & 4 & 2 & 1 \\
\xi_4 \to & 8 & 4 & 2 & 1 & & \\
\xi_4 \to & 8 & 4 & 2 & 1 & & \\
\xi_3 \to & 4 & 2 & 1 & & & \\
\hline
 & 52 & 26 & 13 & 6 & 2 & 1
\end{array}$$

$Sq^{52}\, Sq^{26}\, Sq^{13}\, Sq^{6}\, Sq^{2}\, Sq^{1}$

pairs with $\xi_6\, \xi_4^2\, \xi_3$

$\langle\ ,\ \rangle = 1$

triangular matrix

This proves Milnor's result.

Wednesday, February 12, 2025 9:36 AM

Milnor: $A_* = \mathbb{Z}/2[\xi_1, \xi_2 \cdots]$ $|\xi_n| = 2^n - 1$

$$\psi(\xi_n) = \sum_{k=0}^{n} \xi_k^{2^{n-k}} \otimes \xi_{n-k} \qquad (\xi_0 = 1).$$

Dualizing, we get the Steenrod algebra.

What about $Sq^n Sq^m$ $n < 2m$?

= Adem relations

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{m-k-1}{n-2k} Sq^{m+n-k} Sq^k$$

$m+n-k \geq 2k$

Method of proof: $H^*(\Sigma_4; \mathbb{Z}/2)$

$$\mathbb{Z}/2 \times \mathbb{Z}/2 < \mathbb{Z}/2 \ltimes (\mathbb{Z}/2 \times \mathbb{Z}/2)$$

(May: LNM 168)

<u>Next time</u>: Calculating $\Omega^{\text{unoriented}}$ using the Steenrod algebra (<u>easy</u>!)