Math 696

Friday, February 14, 2025 8:58 AM

Calculate $\pi_* MO = \Omega_*^{\text{unoriented}}$ ← unoriented cobordism groups

Recall that $A_* = H\mathbb{Z}/2_* H\mathbb{Z}/2$ is the dual of the Steenrod algebra over $\mathbb{Z}/2$ $\quad A^* = H\mathbb{Z}/2^* H\mathbb{Z}/2$

A_* is a commutative Hopf algebra, graded (degrees ≥ 0), connected: $A_0 = \mathbb{Z}/2$.

$$A_* = \mathbb{Z}/2[\xi_n \mid n = 1, 2, \dots] \qquad |\xi_n| = 2^n - 1$$

$$\psi(\xi_n) = \sum_{k=0}^{n} \xi_k^{2^{n-k}} \otimes \xi_{n-k} \qquad (\xi_0 = 1)$$

The discussion in Milnor-Stasheff about the connection between Stiefel-Whitney classes and the Steenrod operations:

Thom isomorphism: $H^*(MO; \mathbb{Z}/2) \xrightarrow[\cong]{\Phi} H^*(BO; \mathbb{Z}/2) = \mathbb{Z}/2[w_1, w_2, \dots]$, $\quad w_n \in H^*(BO;\mathbb{Z}/2)$

<u>The recipe</u>: $w_n = \Phi\, Sq^n\, \Phi^{-1}(1)$.

Through $\mathbb{RP}^\infty = BO(1) \to BO$, this simply says that

$$\boxed{H\mathbb{Z}/2_* MO \longrightarrow H\mathbb{Z}/2_* H\mathbb{Z}/2 = A_*}$$

← Milnor's computation of A_*: This map is onto.
(map of algebras, onto the generators)

$$H\mathbb{Z}/2 \wedge MO \xrightarrow{\mathrm{Id} \wedge u} H\mathbb{Z}/2 \wedge H\mathbb{Z}/2$$

$$MO \xrightarrow{u} H\mathbb{Z}/2$$

$$1 \in H\mathbb{Z}/2^0 MO$$

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$$\underbrace{H\mathbb{Z}/2_* MO}_{M} \twoheadrightarrow A_* = H\mathbb{Z}/2_* H\mathbb{Z}/2$$

a homomorphism of comodule algebras

(MO ring spectrum $\Rightarrow H\mathbb{Z}/2_* MO$ is an algebra

comodule $\Leftarrow$ dual of cohomological operations)

Theorem (Milnor-Moore): Let A be a (graded)-commutative (super) (says nothing over $\mathbb{Z}/2$) connected Hopf algebra. Let M be a graded A-comodule algebra finitely generated in ≥ 0 degrees, let

$$M \twoheadrightarrow A$$

be a homomorphism of comodule algebras. Then

$$M \cong \bigoplus_i A[k_i]$$ shift by $k_i \geq 0$

as an A-comodule.

Example: $M = H\mathbb{Z}/2_* MO$

$A = A_* = H\mathbb{Z}/2_* H\mathbb{Z}/2$

Suppose we have the Milnor–Moore theorem.

$$H\mathbb{Z}/2_* MO = \bigoplus_i H\mathbb{Z}/2_* H\mathbb{Z}/2\,[k_i] \quad \text{shift}$$

How can we get to $\pi_* MO$? We have an element $z_i \in H\mathbb{Z}/2_{k_i} MO$ which is primitive $\overset{\text{comodule}}{\Delta}(z_i) = z_i \otimes 1$. But by the UCT, we have some element $\overset{\text{Steenrod}}{\alpha_i} \in H\mathbb{Z}/2^{k_i} MO$, $\langle \alpha_i, z_i \rangle = 1$.

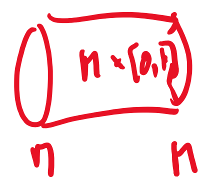
$$\alpha_i : MO = \Sigma^{k_i} H\mathbb{Z}/2$$

So we have constructed a morphism of spectra

$$MO \xrightarrow{\varphi} \bigvee_i \Sigma^{k_i} H\mathbb{Z}/2$$

which induces an isomorphism in $H\mathbb{Z}/2$-homology. We claim that φ also induces an isomorphism in $\underbrace{\text{homotopy groups}}_{\mathbb{Z}/2\text{-modules (from cobordism}}$ ($\Rightarrow$ is an equivalence)

$$MO \sim \bigvee_i \Sigma^{k_i} H\mathbb{Z}/2$$



$2M \sim 0$

unoriented cobordism)

We can use Hurewicz Thm. (The lowest homotopy group of a spectrum bounded below $\cong$ to its $H\mathbb{Z}_*$)

We can now do counting:

$$\mathrm{rank}_{\mathbb{Z}/2} H\mathbb{Z}/2_* MO = \mathrm{rank}_{\mathbb{Z}/2} H\mathbb{Z}/2^* MO < \infty$$

For a graded $\mathbb{Z}/2$-module M, the Poincaré series $P(M) = \sum_k \mathrm{rank}\, M_k \cdot x^k$.

$$P(H\mathbb{Z}/2_* MO) = P(\underbrace{H\mathbb{Z}/2^* MO}_{\overset{\theta}{\cong}\ \mathbb{Z}/2[w_1, w_2, w_3, \dots]})$$

$|w_n| = n$

$$A_* = \mathbb{Z}/2[\xi_1, \xi_2, \dots]$$

$$|\xi_k| = 2^k - 1$$

$$\therefore\ P(\pi_* MO) = \frac{P(H\mathbb{Z}/2^* MO)}{P(A_*)} = P(\mathbb{Z}/2[a_n \mid n \neq 2^k - 1])$$

But we have an injective map

$$\boxed{\mathbb{Z}/2[a_n \mid n \neq 2^k - 1] \to \pi_* MO}$$

Milnor manifold

Both sides have the same Poincaré series

$\therefore$ isomorphism

Next: Complex cobordism $\underbrace{\text{groups}}_{\text{(ring)}}$ $\pi_* MU$.

↖ Chern numbers

Based on what we said:

$$\pi_* MU \xrightarrow[\text{Chern numbers}]{} H\mathbb{Z}_* MU \overset{\alpha}{\cong} H\mathbb{Z}_* BU = \mathbb{Z}[c_1, c_2, \cdots]$$ ✓

There are "Milnor-Johnston manifolds" X_n (smooth proj' varieties over $\mathbb{C}$)

detect indecomposable element in $H\mathbb{Z}_* BU$

$$s_n[X_n] = \begin{cases} 1 & \text{if } n \neq p^k - 1 \\ p & \text{if } n = p^k - 1 \end{cases} \qquad p \text{ prime}$$

$\pi_* MU$ is the Lazard ring L, supporting the universal FGL.

This would lead us to conjecture

$$\pi_* MU = \mathbb{Z}[x_1, x_2, \cdots]. \qquad |x_n| = 2n$$

(Note: FGL $\Rightarrow$ MU cannot be a wedge of shifted copies of $H\mathbb{Z}$.) (in HA A alg. group)

First, we need to discuss the Steenrod algebra at an odd prime:

$$A^* = H\mathbb{Z}/p^* H\mathbb{Z}/p.$$

Formal discussion: Dual

$$A_* = H\mathbb{Z}/p_* H\mathbb{Z}/p.$$

(Milnor: Steenrod algebra and its dual discusses primarily p odd)

A_* is $\underbrace{\text{graded}}_{\text{super}}$ - commutative

Odd degree elements occur: Bockstein $\beta \in A^1$

$$\beta : H\mathbb{Z}/p^k X \to H\mathbb{Z}/p^{k+1} X$$

(say, X a space). On chain level

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

Singular cochain cxs:

$$0 \to C^*(X; \mathbb{Z}/p) \to C^*(X; \mathbb{Z}/p^2) \to C^*(X; \mathbb{Z}/p) \to 0$$

Long exact sequence in cohomology, the connecting map is β.

Examples where it is $\neq 0$

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \to & \mathbb{Z}/p & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \mathbb{Z}/p & \xrightarrow{p} & \mathbb{Z}/p^2 & \to & \mathbb{Z}/p & \to & 0 \end{array}$$

unit sphere in $\mathbb{C}^{\oplus\infty}/\mathbb{Z}/p$

We also have

$$\beta : H^k(X; \mathbb{Z}/p) \to H^{k+1}(X; \mathbb{Z})$$

LES $\Rightarrow$ $|a| = 1$ $|b| = 2$

$$H^*B\mathbb{Z}/p = \Lambda_{\mathbb{Z}/p}[a] \otimes \mathbb{Z}/p[b]$$

$\boxed{\beta(a) = b}$

exercise