Math 696

Monday, February 17, 2025 8:57 AM

Steenrod operations at an odd prime p

Steenrod algebra: $A^* = H\mathbb{Z}/p^* H\mathbb{Z}/p$

$\beta : H^k(X;\mathbb{Z}/p) \longrightarrow H^{k+1}(X;\mathbb{Z}/p)$, X = space or spectrum

<u>Example</u>: $H^*(B\mathbb{Z}/p;\mathbb{Z}/p) = \Lambda_{\mathbb{Z}/p}[a] \otimes \mathbb{Z}/p[b]$, $\beta a = b$

$|a| = 1 \qquad |b| = 2$

What other operations can we construct in $H\mathbb{Z}/p^*$? Following the case of $\mathbb{Z}/2$: For $p>2$, X space

$$C_*X \xrightarrow{\psi} \underbrace{C_*X \otimes \cdots \otimes C_*X}_{p \text{ terms}}$$

Σ_p (symmetric group)-equivariant?

$$\underbrace{C_*E\Sigma_p}_{\text{free } \mathbb{Z}[\Sigma_p]\text{-resolution of } \mathbb{Z}} \otimes C_*X \xrightarrow[\Sigma_p\text{-equiv.}]{} C_*(X)^{\otimes p} \qquad \Big|\ \otimes_{\mathbb{Z}[\Sigma_p]} \mathbb{Z}/p$$

$\circlearrowleft$ permutation of terms

<u>Sign</u> $\quad xy \rightsquigarrow yx$

$(-1)^{|x||y|}$

dualize

$$C_*E\Sigma_p \otimes C^*X^{\otimes p} \xrightarrow{\Sigma_p\text{-equiv.}} C^*X$$

Apply $\otimes_{\mathbb{Z}[\Sigma_p]} \mathbb{Z}/p$:

$$H_*(\Sigma_p, C^*(X)^{\otimes p} \otimes \mathbb{Z}/p) \longrightarrow H^*(X; \mathbb{Z}/p)$$

signed permutations

how to calculate this homology?

$$\|$$

$$H_*(\Sigma_p, \underbrace{H^*(X; \mathbb{Z}/p)^{\otimes p}}_{\text{signed permutation rep.}}) = \left(H_*(\mathbb{Z}/p, H^*(X; \mathbb{Z}/p)^{\otimes p})\right)^{\mathbb{Z}/p^\times}$$

order $p!$

$\mathbb{Z}/p \subset \Sigma_p$ Weyl group $= \mathbb{Z}/p^\times \cong \mathbb{Z}/(p-1)$

Sylow subgroup $N(H)/H$

The operations one gets : P^k , βP^k

$\in H^*(X; \mathbb{Z}/p)$ $|P^k| = 2k(p-1)$ $|\beta| = 1$

$|x|$ even X space

$P^k x$, $k = 0, \cdots, \frac{|x|}{2}$ $|x| = 2k$

$P^0 x = x$, $\underbrace{P^{\frac{|x|}{2}} x = x^p}$ $\underbrace{P^k}_{} x = x^p$

dim: $2k + 2k(p-1) = 2kp$

Milnor's discussion: $H^*(B\mathbb{Z}/p;\mathbb{Z}/p) = \Lambda_{\mathbb{Z}/p}[a] \otimes \mathbb{Z}/p[b]$

$|a| = 1 \quad |b| = 2$

$\beta a = b.$

$k > \frac{|x|}{2}$

$P^k x := 0$

$P(x) = \sum_{k=0}^{\infty} P^k(x)$

$|x|$ even

$P(b) = b + b^p$

$P(b^{p^k}) = b^{p^k} + b^{p^{k+1}}$

A^* is a Hopf algebra, dual: $A_* = H\mathbb{Z}/p_* H\mathbb{Z}/p$.

graded commutative

If X is a space: comodule algebra structure

X f.d.

if not, $\hat{\otimes}$

$$H^*(X;\mathbb{Z}/p) \xrightarrow{\lambda} H^*(X;\mathbb{Z}/p) \otimes_{\mathbb{Z}/p} A_*$$

$X = B\mathbb{Z}/p$:

$$\lambda(a) = a \otimes 1 + \sum_{k\geq 0} b^{p^k} \otimes \tau_k \qquad |\tau_k| = 2p^k - 1$$

$$\lambda(b) = \sum_{k\geq 0} b^{p^k} \otimes \xi_k \qquad |\xi_k| = 2p^k - 2 \quad \Big| \quad \xi_0 = 1.$$

$$(\lambda \otimes 1) \circ \lambda = (1 \otimes \psi) \circ \lambda :$$

<u>Milnor's Theorem</u>: $A_* = \Lambda_{\mathbb{Z}/p}[\tau_0, \tau_1, \tau_2, \dots] \otimes \mathbb{Z}/p[\xi_1, \xi_2, \dots]$

$$|\tau_k| = 2p^k - 1 \qquad |\xi_k| = 2p^k - 2$$

$$\psi(\xi_n) = \sum_{k=0}^{n} \xi_k^{p^{n-k}} \otimes \xi_{n-k} \qquad \xi_0 = 1$$

$$\psi(\tau_n) = \tau_n \otimes 1 + \sum_{k=0}^{n} \xi_k^{p^{n-k}} \otimes \tau_{n-k}.$$

A discussion of $H\mathbb{Z}/p^* K(\mathbb{Z}/p, n)$ similar to before shows that this gives all of A_* (hence A^*). ← Also, Adem relations between βP^k, P^k give an alternative description of A^*

$$P_* = \mathbb{Z}/p[\xi_1, \xi_2, \dots], \quad \psi(\xi_n) = \sum_{k=0}^{n} \xi_k^{p^{n-k}} \otimes \xi_{n-k}$$

is a sub-Hopf algebra of A_*.

(For $p=2$, $\xi_n := \xi_n^2$)

$$P_* \subset A_*$$

What does Milnor-Moore say for $H_*(MU;\mathbb{Z}/p)$?

$MU = \operatorname{colim}_k \Sigma^{-2k} BU(k)^{\gamma^k}$

$H_*(\mathbb{CP}^\infty;\mathbb{Z}/p)[-2] \subset H_*(MU;\mathbb{Z}/p)$

$\Sigma^{\infty-2}\mathbb{CP}^\infty \to MU$

$k=1$

all even, polynomial algebra on generators in degrees $2,4,6,\dots$

generators of $H_{2n}(\mathbb{CP}^\infty;\mathbb{Z}/p)$

$|L|=2$

$H_*(\mathbb{CP}^\infty;\mathbb{Z}/p)$ | H^*

$\cong$ in even degrees $\to$ $\uparrow$

$H_*(B\mathbb{Z}/p;\mathbb{Z}/p)$ | H^*

$b \to b$

$\mathbb{Z}/p \subset S^1$

Comodule algebra structure

$H_*(MU;\mathbb{Z}/p) \to H_*(MU;\mathbb{Z}/p) \otimes A_*$

$\cup$

$H_*(MU;\mathbb{Z}/p) \otimes P_*$

locating

Apply Milnor-Moore to P_*:

$H_*(MU;\mathbb{Z}/p)$, as a P_*-comodule, is $\underbrace{P_*[x_n \mid n \neq p^k-1]}$

A_*

$|x_n| = 2n$

$$H_*(MU; \mathbb{Z}/p) = P_*[x_n \mid n \neq p^h - 1] \qquad |x_n| = 2n$$

A_* - comodule

Is there a spectrum with homology P_* (as an A_* - comodule)?

Yes, Brown-Peterson spectrum BP.

Should we hope that $MU = \bigvee \Sigma^{2k_i} BP$?

Yes, after localization (or completion).

Moore spectrum : A abelian group

$H_0 MA = A$

$H_k MA = 0 \quad k \neq 0$

ab. groups

$$0 \to \mathbb{Z}T_1 \hookrightarrow \mathbb{Z}T_2 \to A \to 0$$

$$\bigvee_{T_1} S \to \bigvee_{T_2} S \to MA$$

$\simeq$ mapping cone.

If X is cell spectrum with bounded below cells, finitely many cells in each degree "finite type"

$$X_{(p)} = X \wedge M\mathbb{Z}_{(p)} \quad ; \quad \hat{X}_p = H \wedge M\mathbb{Z}_p.$$

We will be able to show that

$$MU^\wedge_p = \bigvee BP^\wedge_p [2k_i]$$

$$MU_{(p)} = \bigvee BP_{(p)} [2k_i]$$

The splitting suggested by the A_*-comodule structure on homology is true after localization or completion

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The Adams spectral sequence:

$\pi_* X^\wedge_p$ from $H_*(X; \mathbb{Z}/p)$ as a A_*-comodule

$\Leftrightarrow H^*(X; \mathbb{Z}/p)$ as A^*-module

when X bounded below finite type.