Adams spectral sequence

let X be a spectrum of finite type bounded below
(cell spectrum, dimension of cells bounded below, finitely many cells in each dimension)

? $\pi_* X$ are bounded below, finitely generated abelian groups

localization and completion in this case:

$$X_{(p)} = X \wedge M\mathbb{Z}_{(p)} \qquad \pi_* X_{(p)} = (\pi_* X) \otimes \mathbb{Z}_{(p)} = \pi_* X_{(p)}$$

$$X^\wedge_p = X \wedge M\mathbb{Z}_p \qquad \pi_* X^\wedge_p = (\pi_* X) \otimes \mathbb{Z}_p = \pi_* X^\wedge_p$$

knowing this for all p recovers $\pi_* X$

Adams spectral sequence: X as above

graded

$$E_2^{s,t} = \mathrm{Ext}^s_{A^*}(H\mathbb{Z}/p^* X, \mathbb{Z}/(p))_t \Rightarrow \pi_{t-s} X^\wedge_p$$

steenrod algebra
at a fixed prime p

Where the Adams spectral sequence comes from:

Adams resolution:

homotopy cofiber

$$X =: X_0$$

$$X_0 \xrightarrow{\mathrm{Id}_{X_0} \wedge (\eta : S \to H\mathbb{Z}/p)} H\mathbb{Z}/p \wedge X_0 \longrightarrow X_1$$

$$\vdots$$

$$X_s \to H\mathbb{Z}/p \wedge X_s \to X_{s+1}$$

$$\vdots$$

We get an exact couple by applying π_t:

$$\underbrace{\pi_t X_{s+1}}_{D_1^{s+1,t-s-1}} \to \underbrace{\pi_t X_s}_{D_1^{s,t-s}} \longrightarrow \underbrace{H\mathbb{Z}/p_t X_s}_{\substack{E \\ E_1^{s,t-s}}} \to \underbrace{\pi_t X_{s+1}}_{D} \longrightarrow \underbrace{\pi_{t-1} X_s}_{D} \to \cdots$$

grade cohomologically

$$\begin{tikzcd} D_1 \arrow[r] & D_1 \arrow[dl] \\ E_1 \arrow[u] & \end{tikzcd}$$

cohomological exact couple

Identifying the E_2-term: Apply $H\mathbb{Z}/p^*$ instead of π_t to

$$X_s \to H\mathbb{Z}/p \wedge X_s \to X_{s+1}$$

$$0 \leftarrow H^* X_s \leftarrow A^* \otimes H^* X_s \longleftarrow H^* X_{s+1} \leftarrow 0$$

$$F(H\mathbb{Z}/p \wedge X_s, H\mathbb{Z}/p[?])$$

$$F(H\mathbb{Z}/p, H\mathbb{Z}/p[?]) \otimes F(X_s, H\mathbb{Z}/p)$$

Künneth thm. in cohomology

(Finite type: do it in homology, then UCT)

We get an exact sequence:

$$0 \leftarrow H\mathbb{Z}/p^* \underbrace{X}_{X_0} \leftarrow \underbrace{A^* \otimes H\mathbb{Z}/p^* X_0 \xleftarrow{\partial} A^* \otimes H\mathbb{Z}/p^* X_1 \xleftarrow{\partial} A^* \otimes H\mathbb{Z}/p^* X_2 \leftarrow \cdots}_{\text{A free } A^*\text{-resolution of } H\mathbb{Z}/p^* X}$$

What does it have to do with (E_1, d)?

$$E_1^{s,t} = H\mathbb{Z}/p_* X_s = \mathrm{Hom}_{A^*}(A^* \otimes H\mathbb{Z}/p^* X_s, \mathbb{Z}/p)$$

$$d_1 = \mathrm{Hom}_{A^*}(\partial, \mathbb{Z}/p) \qquad H\mathbb{Z}/p_* X_s \to H\mathbb{Z}/p_* X_{s+1} \qquad = \mathrm{Hom}_{\mathbb{Z}/p}(H\mathbb{Z}/p^* X_s, \mathbb{Z}/p)$$

<u>Definition of</u> Ext $\Rightarrow$ identification of E_2-term.

What about convergence? Algebraic discussion = same as for all spectral sequences.

Topological discussion:

$$\Sigma^{-1} X_{s+1} \xrightarrow{\text{connecting map}} X_s \to H\mathbb{Z}/p \wedge X_s \to X_{s+1}$$

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"Adams tower":

Statement needed:

$\simeq$ in D(spectra)

$$\mathrm{holim}\left(\cdots \to \Sigma^{-3}X_3 \to \Sigma^{-2}X_2 \to \Sigma^{-1}X_1 \to X_0 = X\right) \simeq *$$

One feature worth mentioning: Enough to do this for HA where A is a finitely generated abelian group.

This is because of the Postnikov tower: If X is a spectrum of finite type bounded below, the lowest non-zero homotopy group is $\pi_s X$. By the Hurewicz theorem, we have the characteristic map

$$X' \to X \to H\pi_s X[s]$$

shift up by s

homotopy fiber:

$\pi_k X' = \pi_k X$ for $k \geq s+1$

$\pi_k X' = 0$ for $k \leq s$

$X' = X^{(1)}$

Chapter 2 of Ravenel: Complex cobordism and stable homotopy grps.

direct computation — Exercise

$X = X^{(0)}$

Iterate. $\mathrm{holim}\left(\cdots \to X^{(k)} \to X^{(k-1)} \to \cdots \to X^{(0)}\right) \simeq *$

Combine the Adams and Postnikov towers, reduce the problem to $H\mathbb{Z}_p$, $H\mathbb{Z}/p^r$. $\square$

We calculate $\pi_* MU$ ($= \Omega^{\text{complex}}$) using the Adams spectral sequence.

Recall the Steenrod algebra: $p > 2$ $\quad \Lambda[\tau_0, \tau_1, \tau_2, \dots] \otimes \mathbb{Z}/p[\xi_1, \xi_2, \dots]$

$$|\tau_n| = 2p^n - 1$$
$$|\xi_n| = 2p^n - 2$$

$\psi = \dots$

$\boxed{P_* = \mathbb{Z}/p[\xi_1, \xi_2, \xi_3, \dots] \text{ is a sub-Hopf algebra}}$

$p = 2$ $\quad \mathbb{Z}/2[\xi_1, \xi_2, \dots]$

$$|\xi_n| = 2^n - 1$$

$\xi_n = \tau_{n-1}$

Put $\xi_n := \xi_n^2$

(Super)-commutative Hopf algebra is the coordinate ring of a (super)-group scheme $\quad A_* = \mathcal{O}_{\mathcal{A}}$, $\quad P_* = \mathcal{O}_{\mathcal{P}}$

Morphism of super-group schemes

$$\mathcal{K} \triangleleft \mathcal{A} \twoheadrightarrow \mathcal{P}$$

We always have a kernel

$$\mathcal{O}_{\mathcal{K}} = A_* \otimes_{P_*} \mathbb{Z}/p = \Lambda[\tau_0, \tau_1, \dots]$$

In $\mathcal{O}_{\mathcal{K}}$, τ_n are primitive

Dualise: We have a homomorphism of non-commutative algebras

$$A^* \twoheadrightarrow P^*$$

$|Q_n| = 2p^n - 1$ are called the Milnor primitives

The kernel is the exterior algebra $\mathcal{Q}_X^\vee = \Lambda[Q_0, Q_1, \dots]$

can be chosen to be primitive

Exercise (a dual of a primitively generated exterior Hopf algebra is primitively generated exterior)

$\Lambda[Q_0, Q_1, \dots] \subset A^*$ ← Steenrod algebra for all primes p.

subalgebra

For $\pi_* MU$, it turns out we only need to do $\mathrm{Ext}_{\Lambda[Q_0, Q_1, \dots]}(\mathbb{Z}/p, \mathbb{Z}/p)$.

We wrap this up next time.