Math 696

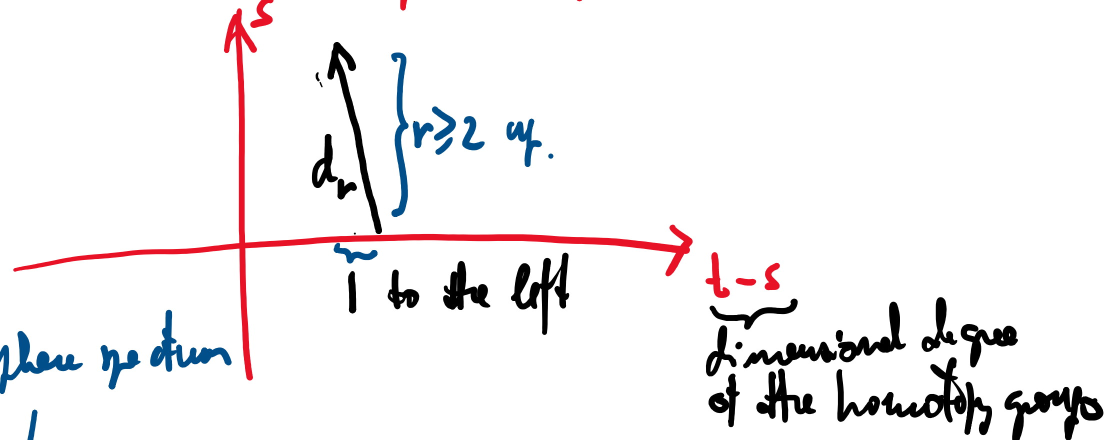
Friday, February 21, 2025 8:58 AM

Adams spectral sequence $\Rightarrow \pi_* MU_p^\wedge$

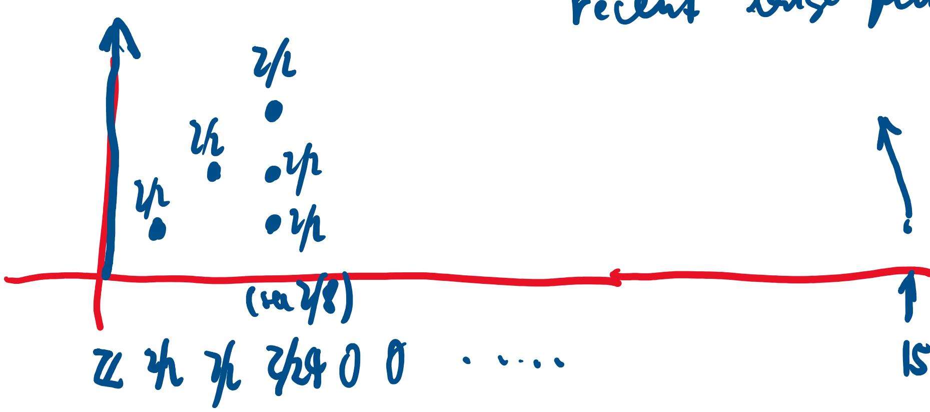
X bounded below spectrum of finite type:

$$E_2^{s,-t} = \mathrm{Ext}^s_{A^*}(H\mathbb{Z}/p^* X, \mathbb{Z}/p)_t \Rightarrow \pi_{t-s} X_p^\wedge$$

Usually, we draw the Adams spectral sequence:



Example: $\pi_* S_2^\wedge$ } stable 2-stems   Xu & collaborators recent large progress



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$\pi_* MU^{\wedge}_p$ is easier than $\pi_* S$.

Milnor-Moore: $H\mathbb{Z}/p_* MU = P_*[x_n \mid n = p^k - 1]$

$|x_n| = 2$

as a comodule over A_* polynomial ring

$p > 2$

$A_* = \Lambda[\tau_0, \tau_1, \cdots] \otimes \mathbb{Z}/p[\xi_1, \xi_2, \cdots]$ $\psi \ldots$

$|\tau_k| = 2p^k - 1$ $|\xi_k| = 2p^k - 2$

$A_* \supset P_* = \mathbb{Z}/p[\xi_1, \xi_2, \ldots]$

"Coordinate rings of (algebraic) unipotent group-schemes"

$P_* \subseteq A_* \longrightarrow \Lambda_*$ $\psi(\tau_k) = 1 \otimes \tau_k + \tau_k \otimes 1$

$\Lambda_* = \Lambda[\tau_0, \tau_1, \ldots]$

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Dualizing: $\Lambda[Q_0, Q_1, \dots] \to A^* \longrightarrow P^*$

$\quad \| $

$\quad \Lambda^*$

$\psi(Q_i) =$

$= Q_i \otimes 1 + 1 \otimes Q_i$ ← Q_i = Milnor primitives

$|Q_i| = 2p^i - 1$

E_2-term of the Adams spectral sequence for $\pi_* MU^\wedge_p$:

$$\mathrm{Ext}_{A^*}(H\mathbb{Z}/p^* MU, \mathbb{Z}/p) = \mathrm{Ext}_{A^*}\Big(\underbrace{\bigoplus_{\text{even shifts}}}_{\text{Poincaré series of } \mathbb{F}_p[x_i \mid i \neq p^k - 1]} P^*, \mathbb{Z}/p\Big)$$

? $\mathrm{Ext}_{A^*}(\underset{\sim}{P^*}, \mathbb{Z}/p) = \mathrm{Ext}_{A^*}(A^* \otimes_{\Lambda^*} \mathbb{Z}/p, \mathbb{Z}/p) = \mathrm{Ext}_{\Lambda^*}(\mathbb{Z}/p, \mathbb{Z}/p)$

Exercise: Λ^*-coalg. of $\mathbb{Z}/p$ $A \otimes_{\Lambda^*}$

$P^* = A^* \otimes_\Lambda \mathbb{Z}/p$

dual: $P_* = \{x \in A_* \mid \psi(x) \text{ has no } \tau_i\text{'s}\}$

$\Lambda^* = \Lambda[Q_0, Q_1, \dots]$

$|Q_k| = 2p^k - 1 > 0$

Künneth

$$\mathrm{Ext}_{A^*}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathrm{Ext}_{\underbrace{\Lambda[Q_0, Q_1, \dots]}_{\Lambda[Q_0] \otimes \Lambda[Q_1] \otimes \cdots}}(\mathbb{Z}/p, \mathbb{Z}/p) = \bigotimes_{n \geq 0} \underbrace{\mathrm{Ext}_{\Lambda[Q_n]}(\mathbb{Z}/p, \mathbb{Z}/p)}_{\mathbb{Z}/p[v_n]}$$

s t

$|v_n| = (1, 2p^n - 1)$

$t-s$ total $|v_n| = 2p^n - 2$

$$\mathrm{Ext}_{A^*}(\mathbb{F}_p^*, \mathbb{Z}/p) = \mathbb{Z}/p[v_0, v_1, v_2, \cdots] \qquad |v_k| = \underbrace{(1, 2p^k - 1)}_{\text{total}: 2p^k - 2}$$

$$\mathrm{Ext}_{A^*}(H\mathbb{Z}/p^* MU, \mathbb{Z}/p) = \mathbb{Z}/p[v_0, v_1, v_2, \cdots] \otimes \mathbb{Z}/p[x_n \mid n \neq p^k - 1]$$

even

all total degrees are even, so Adams spectral sequence collapses.

$$|x_n| = \underbrace{(0, 2n)}_{\text{total}: 2n}$$

$|v_0| = 0$ v_0 represents p

$Q_0 = \beta$ (the generator of $H\mathbb{Z}/p^1 H\mathbb{Z}/p = A^1$).

So

$$\pi_* MU^\wedge_p = \mathbb{Z}_p[x_n \mid n \neq p^k - 1] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[v_1, v_2, v_3, \dots]$$

$|x_n| = 2n$ $\qquad\qquad$ $|v_k| = 2p^k - 2$

one polynomial generator in every even degree > 0.

What about $\pi_* MU$? Thm MU f.g. (MU finite type bounded below)

$$\pi_* MU = \mathbb{Z}[x_n \mid n = 1, 2, \dots]$$

$$|x_n| = 2n$$

in terms of complex cobordism, encodes all Chern numbers

The Hurewicz homomorphism

$$\pi_* MU \xrightarrow{h} H\mathbb{Z}_* MU$$

$$\mathbb{Z}[x_n \mid n = 1, 2, \dots] \qquad \mathbb{Z}[m_n \mid n = 1, 2, \dots]$$

Modulo decomposables $(m_1, m_2, \dots)^2$

$x_n \longmapsto m_n$ $\quad n \neq p^k - 1$ to prime

$x_n \longmapsto p\, m_n$ $\quad n = p^k - 1$

Recall FCL theory: Every complex line bundle is MU-orientable

have MU^*-valued Thom iso.

(∴ every complex bundle by Schubert calculus)

complex oriented spectrum

$[\Sigma^{-2}\mathbb{CP}^\infty \longrightarrow MU]$

$\cong (\mathbb{CP}^\infty)^{\gamma_G^1}$

$$MU^*\mathbb{CP}^\infty = MU^*[[x]]$$

$1+1=2$

$x = c_1(\gamma_G^1)$

MU_{-*} degree ≤ 0

L, M cx. line bundles

$$c_1^{MU}(L \otimes M) = c_1^{MU}(L) +_F c_1^{MU}(M)$$

formal group law

$x +_F y$: power series

$x +_F 0 = x$

$x +_F y = y +_F x$

$x +_F (y +_F z) = (x +_F y) +_F z.$

F_{MU} is the universal formal group law.

Formal group laws on a (comm.) ring R:

Morphisms of rings $L \to R$

$L = \mathbb{Z}[a_{ij}] /$ relations following from F being an FGL

$x +_F y = \sum a_{ij} x^i y^j$ } We proved $\underbrace{\pi_* MU = L}$

Strict isomorphism (reparametrization) of FGL:

$F \to G$

is a power series $h(x) = x + b_1 x^2 + b_2 x^3 + \cdots$

intertwining relation:

$$h(x) +_G h(y) = h(x +_F y).$$

We proved that if R is any ACU ring, then over $R \otimes \mathbb{Q}$, every formal group law F is strictly $\cong$ to $+$: $x+y$

$\log_F x : F \to +$ $\qquad H\mathbb{Z}_* MU = \mathbb{Z}[m_1, m_2, \cdots]$

For the universal FGL F, $\log_F x = x + m_1 x^2 + m_2 x^3 + \cdots$

We also showed that

$$L \Big/ \begin{pmatrix} a_{ij} \mid i+j \neq n+1 \end{pmatrix} = \mathbb{Z}[x_n]$$
$$(a_{ij})^2$$

all

$$\log x_n = \text{unit} \cdot \gcd\left\{ \binom{n+1}{i} \;\middle|\; 0 < i < n+1 \right\}$$
$$= 1 \qquad n+1 \neq p^k$$
$$p \qquad n+1 = p^k$$

Ravenel: Complex cobordism ... Appendix A2

Next: A little bit more on FGL $\rightsquigarrow$ a lot of interesting generalized cohomology theories

"chromatic" homotopy theory

relations with number theory

I will talk about final projects.