Posted on Canvas: Possible topics for final project

The method of formal group laws:

- R commutative ring
  $x +_F y \in R[[x,y]]$

$$\left.\begin{aligned} x +_F 0 = 0 +_F x &= x \\ x +_F y &= y +_F x \\ x +_F (y +_F z) &= (x +_F y) +_F z \end{aligned}\right\} \text{FGL}$$

The Lazard ring:

$$L = \mathbb{Z}[a_{i,j}] / \text{relations making } x +_F y \text{ an FGL} \qquad a_{i,j} = a_{j,i}$$

write $x +_F y = \sum a_{i,j} \cdot x^i y^j$. $\{\text{Morphisms of rings } L \to R\} \cong \text{FGL's on } R$

$\underbrace{L \to R}_{\text{substitute for } a_{i,j}}$

Theorem: $L \cong \pi_* MU = \Omega^{\text{complex}}$.

In the process: $L = \mathbb{Z}[x_1, x_2, x_3, \dots]$

On $R \otimes \mathbb{Q}$, any two FGL's F, G are strictly isomorphic.

There exists a series $h(x) = x + m_1 x^2 + m_2 x^3 + \cdots$

$$h(x) +_G h(y) = h(x +_F y).$$

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$$\log_F x : F \xrightarrow{\cong} + \qquad \text{works over } R\otimes\mathbb{Q}$$

$x+y$

$\log_F x = x + m_1 x^2 + m_2 x^3 + \cdots$

$\exp_F x$ formal inverse $\Big(\underbrace{\text{Lagrange inversion formula}}\Big)$

$$\mathbb{Z}[x_1, x_2, x_3, \dots] \xrightarrow{c} \mathbb{Z}[m_1, m_2, \dots]$$

$x_n \longmapsto m_n \quad n \neq p^k-1 \quad \bmod (m_1, m_2, \dots)^2$

$p m_n \quad n = p^k - 1$

If E is any complex-oriented spectrum: We can calculate

E-valued Chern classes

$$E^* MU \cong E^* BU = E^*[[c_1, c_2, \dots]]$$

Thom iso — Schubert calculus

(in particular, we can take $E = MU$). If x is a complex orientation of E

$$\text{lift } E^2\mathbb{CP}^\infty \to \underbrace{\widetilde{E}^2\mathbb{CP}^1}_{\ni 1}$$

an arbitrary complex orientation can be written as:

$$x + c_1 x^2 + c_2 x^3 + \cdots$$

(Whitney formula) in $\mathcal{D}$ Spectra

$\therefore$ Any two FGL's on $E^{\text{even}*}$ are strictly isomorphic "up to homotopy"

$\therefore$ Complex orientations on E correspond bijectively to spectral ring maps $MU \to E$.

Applying this to $E = MU$: Giving any strict isomorphism

$F \to G$

arbitrary series $x + c_1 x^2 + c_2 x^3 + \dots\dots$

universal formal group law

the complex orientation of MU

is equivalent to giving a map of ring spectra $MU \to MU$.
(in DSpectra)

If we specify a prime p, when we computed $\pi_* MU_p^\wedge$ (or $\pi_* MU_{(p)}$), we had these distinguished element v_n of dim $2(p^n-1)$

Came from Ext^1 in the Adams spectral sequence.

Formal group laws over $R \otimes \mathbb{Z}_{(p)} = R[n^{-1} \mid p \nmid n]$:

There is a way to get rid of all but logarithmically many coefficients of an FGL.

An FGL F is called <u>p-typical</u> if (over a torsion-free ring)

$$\log_F x = x + \ell_1 x^p + \ell_2 x^{p^2} + \dots + \ell_n x^{p^n} + \dots$$

<u>Theorem:</u>

Ravenel: Complex cobordism... App. A2

Over a $\mathbb{Z}_{(p)}$-algebra R, every FGL is isomorphic to a p-typical FGL. (If R torsion free, the isomorphism is unique)

Proof sketch (torsion-free case): let $\log_F x = x + m_1 x^2 + \dots$

$$\log_G x = x + m_{p-1} x^p + m_{p^2-1} x^{p^2} + \dots$$

same m's

The isomorphism is $\underbrace{\log_G^{-1}}_{\exp_G x}(\log_F x)$

coefficients have no p denominators.

$$\begin{tikzcd} F \arrow[rr, dashed] \arrow[dr, "\log_F x"'] & & G \arrow[dl, "\log_G x"] \\ & t & \end{tikzcd}$$

$\square$

So let's apply this to the localization $MU_{(p)} = MU \wedge M\mathbb{Z}_{(p)}$.

$= (h)\underrightarrow{\mathrm{colim}}\ MU \to MU \to \cdots$

all integers not divisible by p

There exists a ring map

$$MU_{(p)} \xrightarrow{e} MU_{(p)}$$

which corresponds to the strict isomorphism $F \xrightarrow{e} G$

universal property of $M\mathbb{Z}_{(p)}$

universal FGL

p-typical FGL

By the uniqueness of the isomorphism, e is idempotent: $e^2 = e$

algebraically, up to homotopy in case of spectra

$e^2 \simeq e$:

$$(h)\underrightarrow{\mathrm{colim}}\left(MU_{(p)} \xrightarrow{e} MU_{(p)} \xrightarrow{e} \cdots\right) =: eMU_{(p)}$$

$$e\pi_* MU_{(p)} = \pi_* eMU_{(p)} = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \cdots]$$

Cartier ring

$$\downarrow$$

$$\mathbb{Z}_{(p)}[\ell_1, \ell_2, \ell_3, \cdots]$$

coefficients of the logarithm of universal p-typical FGL

$v_k \mapsto p\ell_k \mod (\ell_1, \ell_2, \ldots)$

In homotopy theory, $\underbrace{|v_k| = 2(p^k - 1)}$.

The spectrum $eTMU_{(p)}$ is denoted by BP

(the Brown-Peterson spectrum)

Brown-Peterson 1966

$\pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$

$H\mathbb{Z}/p_* BP = P_*$

[∃ direct construction of BP

Spanier: Spectral sequence ~~ obstruction theory)

Adams spectral sequence obstruction thy

? What is the meaning of v_n in geometry and physics?

(One problem: $|v_n| = 2(p^n - 1)$, depends on prime p)

Nevertheless, v_1 is related to Bott periodicity.

Speculation: $\left.\begin{matrix} v_2 \\ v_1 \end{matrix}\right\}$ is related to Conformal field theory, analysis on loop space.

But we do know a relationship between v_n and

number theory } Lubin-Tate FGL's
p-adic geometry } local class field theory.

We can construct generalized complex-oriented cohomology theories with the Lubin-Tate FGL's.

building blocks of stable homotopy groups of spheres.