What spectra can we make from MU?

$$\pi_* MU = \mathbb{Z}[x_1, x_2, \dots] \qquad |x_n| = 2n$$

a nice ring

Commutative algebra: R commu. ring, $(a_1, a_2, \dots)$ is a <u>regular sequence</u> when a_n is not a $\underbrace{\text{zero divisor}}_{a \neq 0\ \exists b \neq 0\ a \cdot b = 0}$ in $R/(a_1, \dots a_{n-1})$. SES:

$$0 \to R/(a_1, \dots a_{n-1}) \xrightarrow{a_n} R/(a_1, \dots a_{n-1}) \to R/(a_1, \dots a_{n-1}) \to 0.$$

Can we mimic (interpret) such a construction in spectra?

R would have to be a "commutative ring." What does that mean?

Naive interpretation: in the derived category D Spectra, i.e. "up to homotopy".

commutative associative ring spectrum.

You can similarly define module spectrum (diagrams in D Spectra) but it doesn't work: If M is an R-module spectrum, $a \in \pi_* R$, we can make $a: M \to M$ $\quad M \xrightarrow{a \wedge \mathrm{Id}} R \wedge M \xrightarrow{\text{mod.}} M \to C$

not necessarily an R-Mod. spectrum

We need some technique to get past that. In order to talk about a strict commutative monoid, we need a $\underbrace{\text{strict monoidal}}_{\text{in Spectra, not hSpectra}}$ structure.

$\wedge$ depended on choices (does not satisfy that)

There are ways to remedy that. (S-modules, symmetric spectra)

operads

The goal with regard to MU: Take a regular sequence $(a_1, a_2, \dots)$ in $\pi_* MU = \mathbb{Z}[x_1, x_2, \dots]$, and create a spectrum $MU/(a_1, a_2, \dots)$.

Another major commutative algebra construction is localization: $f^{-1}M$, $f \in \pi_* R$.

That is OK:

$$f^{-1}M := \operatorname{hcolim}\left(M \xrightarrow{f} M \xrightarrow{f} M \to \cdots\right)$$

MU does have the structure of a commutative monoid w.r.t. to a symm. monoidal version of $\wedge$. (E_∞-ring spectrum). This can be done.

In the case of MU, there is also an alternative ("cobordism with singularities")

Baas-Sullivan

We will be interested in some of these spectra created from MU:

Examples: $\underbrace{\underbrace{x_1^{-1} MU/(x_n \mid n > 1)}_{\text{coeff: } \mathbb{Z}[x_1]}}_{\text{coeff. } \mathbb{Z}[x_1, x_1^{-1}]} = K$ ← Conner-Floyd.

$|x_1| = 2$

Last time: $BP = e\,MU_{(p)}$ ← Quillen idempotent
p-typical reparametrization of universal FGL

$\pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ $|v_i| = 2(p^i - 1)$.

What do these generators mean? I can also do this integrally.

$$BP := MU/(x_i \mid i \neq p^k - 1 \text{ for any } k)$$

↑ depends on p

$$\pi_* BP = \mathbb{Z}[v_1, v_2, \dots]$$

v_1 is still related to K-theory: Apply Quillen idempotent to K-theory localized at p.

(make multiplicative FGL p-typical).

$$\ell = eK_{(p)} = \mathbb{Z}_{(p)}[v_1, v_1^{-1}]$$

$|v_1| = 2(p-1)$

Adams: $K_{(p)} = \bigvee_{i=1}^{p-1} \ell[2i]$

If v_1 is related to K-theory, what are $v_2, v_3, \ldots$ related to?

One thing that has been studied is called Morava K(n)-theories

$$K(n) = v_n^{-1} BP / (v_i \mid i \neq n)$$

$$\pi_* K(n) = \mathbb{Z}/p\,[v_n, v_n^{-1}]$$

$|v_n| = 2(p^n - 1)$.

"chromatic" homotopy theory

We know little about what these periodicities mean geometrically.

Related to some families of elements in $\pi_* S$.

There are also non-torsion variants of these theories.

Lubin-Tate FGL (local class field theory).

Interpretation via FGL is more successful.

Can come from algebraic groups

The multiplicative group $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$

Elliptic curves "/ $\mathbb{Z}$" $\leftarrow$ Elliptic cohomology

bad primes

? index theorem on loop space?

Witten genus

? Related to conformal field thy (variant: Topological modular forms TMF)

A few hint about the construction methods:

Operad: Think of an $\underbrace{\text{algebraic structure}}$

operations of finite arity
relations: equations

(commutative) monoid:

1 0-ary
$\cdot$ binary

$x \to$ $1 \cdot x = x$
$x \to$ $x \cdot 1 = x$
$\{x,y,z\} \to$ $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$\{x,y\} \to$ $(x \cdot y = y \cdot x)$

these relations are operadic: each of the relations has a set of variables which occur on both sides exactly once

Operadic algebraic structure $\rightsquigarrow$ operad $(\mathcal{P}(n))_{n \geq 0}$

Commutative monoids: $\mathcal{M}(n) = *$

Monoids : $\mathcal{M}(n) = \Sigma_n$

$n!$ element

$\mathcal{P}(n) = \{$all non-equivalent words I can make in $x_1, \dots, x_n$, using each exactly once$\}$

Data of an operad:

Σ_n act on $\mathcal{C}(n)$ (switching variables)

$1 \in \mathcal{C}(1)$

substitution

$$\gamma : \mathcal{C}(n) \times \mathcal{C}(k_1) \times \cdots \times \mathcal{C}(k_n) \to \mathcal{C}(k_1 + \cdots + k_n)$$

Axioms: Associativity, comm., ...

May approach to coherent structures up to homotopy: Replace the operad defining some operadic structure by some other operad $\mathcal{D}$

$$\mathcal{D} \longrightarrow \mathcal{C}$$

homomorphism of operads

where

$$\mathcal{D}(n) \xrightarrow{\simeq} \mathcal{C}(n) \quad \text{is a homotopical equivalence.}$$

Consider $\mathcal{D}$-algebras instead of $\mathcal{C}$-algebras.

Context: Spaces, chain complexes, Spectra

Some of these structures come up in Math. Physics.